

# The unitary dynamical state-locking process, the HSSS quantum search process, and the quantum-computing speedup theory

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## An extended abstract

A unitary dynamical state-locking (UNIDYSLOCK) process is a unitary process that transforms simultaneously two or more orthogonal quantum states to their corresponding non-orthogonal states whose differences may be arbitrarily small. Its inverse process, i.e., the quantum-state-difference amplification (QUANSDAM) process could be used to realize an exponential quantum-computing speedup in solving an unstructured search problem in the quantum-computing speedup theory (X. Miao, arXiv: quant-ph/1105.3573 (2011)). In this paper the principle and mechanism of how a UNIDYSLOCK process and a QUANSDAM process work in a quantum system are described in detail on the basis of the quantum-computing speedup theory. A UNIDYSLOCK process is a unitary quantum dynamical process and also obeys the mathematic-logical principle of the search problem. In pure quantum mechanics ( $QM$ ) a UNIDYSLOCK process does not exist at all, because any  $QM$  unitary operator is not able to change the quantum-state difference between a pair of quantum states, as shown in the paper. It must be comprehended from the viewpoint of the mathematical-logical principle. This unique property of a UNIDYSLOCK process reflects the fundamental importance of the interaction between the quantum physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) and the mathematical-logical principle that a computational problem obeys in the quantum-computing speedup mechanism (or theory). The reversible functional operation of the search problem is originally responsible for the quantum-state-difference change in a UNIDYSLOCK (or QUANSDAM) process. But this interaction leads to that though a  $QM$  unitary operator itself cannot change the quantum-state difference, it could promote a UNIDYSLOCK (or QUANSDAM) process to change greatly the quantum-state difference. It is also shown that any quantum-state effects can not cause the quantum-state-difference change during a UNIDYSLOCK (or QUANSDAM) process in a quantum system, here the quantum-state effects include the quantum-state superposition, coherence interference, entanglement and nonlocal effect, correlation and so on. This conclusion is universal in the frame of unitary quantum dynamics for any quantum computational process that employs a UNIDYSLOCK (or

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QUANSDAM) process to realize its quantum-computing speedup. A UNIDYSLOCK (or QUANSDAM) process is the characteristic quantum computational process (or sub-process) of the quantum-computing speedup theory. Even if a conventional quantum computation that is reversible or unitary contained a UNIDYSLOCK (or QUANSDAM) process, the contribution of the UNIDYSLOCK (or QUANSDAM) process to the quantum-computing speedup of the conventional quantum computation would be secondary or negligible. The reason for this is that if the contribution of the UNIDYSLOCK (or QUANSDAM) process were dominating, then the contribution of the quantum-state effects to the quantum-computing speedup would be secondary or negligible. Generally the quantum-state effects are considered to be essentially responsible for the quantum-computing speedup in the conventional quantum computational theory based on the quantum parallel principle. Therefore, whether or not a UNIDYSLOCK (or QUANSDAM) process can make an essential contribution to a quantum-computing speedup can distinguish the quantum-computing speedup theory from any conventional quantum computational theory. A QUANSDAM process is the second step of the two consecutive steps of the *HSSS* quantum search process, while the first step is the search-space dynamical reduction. The *HSSS* quantum search process is used to solve an unstructured search problem, which is essentially different from a conventional quantum search algorithm. Its quantum-computing speedup mechanism is studied in the paper. Its quantum-computing speedup is original from the interaction mentioned above and hence the interaction is also responsible for the search-space dynamical reduction. In particular, the fundamental quantum-computing resource, i.e., the symmetric structure of the Hilbert space of a composite quantum system is necessary to realize the search-space dynamical reduction. The *HSSS* quantum search process works in both the Hilbert space of the quantum system and the math Hilbert space of the search problem. In contrast, a conventional quantum search algorithm works only in the Hilbert space of the quantum system. The math Hilbert space of a computational problem is a fundamental concept in the quantum-computing speedup theory. It is mathematical and does not have any physical meaning. No concept of the math Hilbert space exists in conventional quantum computation. Due to the math Hilbert space the manner to realize a reversible functional operation in the quantum-computing speedup theory is essentially different from the one in conventional quantum computation. A mathematical-parallel functional operation is the characteristic manner in the quantum-computing speedup theory. In contrast, a quantum-parallel functional operation is the characteristic manner in the conventional quantum computational theory. Due to the interaction between the math Hilbert space and the Hilbert space a reversible functional operation is able to change the quantum-state difference. This leads to that there is a UNIDYSLOCK (or QUANSDAM) process in the quantum-computing speedup theory.

# 1. Introduction

According to the quantum-computing speedup theory [1] the unitary quantum dynamics is considered as the universal quantum driving force to speed up a quantum computation, while the symmetric structure of the Hilbert space of a composite quantum system is the fundamental quantum-computing resource.<sup>2</sup> Here the Hilbert-space symmetric structure is not owned by classical computation and hence it is responsible for an exponential quantum-computing speedup. Both the unitary quantum dynamics and the fundamental quantum-computing resource have been thought of as the two pillars to build an efficient quantum search process (See the early author's works [5, 12]). The interaction between the quantum-physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) and the mathematical-logical principle obeyed by a computational problem to be solved [5, 6, 1, 2] is really fundamentally important to realize an exponential quantum-computing speedup in solving the (hard) computational problem and understand the mechanism of the exponential quantum-computing speedup in quantum computation. The theory proposes a two-step quantum-computing process to solve efficiently an unstructured search problem, which may be called the unstructured quantum search process based on the unitary quantum dynamics and the *Hilbert-space symmetric structure* or briefly the *HSSS* quantum search process. The first step of the *HSSS* quantum search process is an efficient search-space dynamical reduction that the exponentially large unstructured search space of the search problem is reduced dynamically to a polynomially small subspace. In this step the unstructured search space is eliminated and at the same time a polynomially small subspace is generated that carries the information of the component state of the solution state of the search problem<sup>3</sup>. This step is performed in the frame of unitary quantum dynamics and with the help of the Hilbert-space symmetric structure of the quantum system. Its detailed mechanism has been described in Refs. [1, 2] and its efficient realization in an  $n$ -qubit quantum system is described in Ref. [2]. The second step of the process is involved in the *unitary dynamical state-locking (UNIDY SLOCK)* process and its inverse process, i.e., the *quantum-state-difference amplification (QUANS DAM)* process. It mainly employs a QUANS DAM process to extract efficiently the information of all the component states of the solution state. The solution to the search problem then is obtained completely from this solution information. The present work is for

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<sup>2</sup>Tensor product of the Hilbert space of a composite quantum system could lead to that the energy used by a quantum computation in the quantum system may not be exponentially large, here the energy is governed by the tensor-product Hamiltonian of the quantum system. This property of tensor product could be important for a quantum computation, but it may not be related to the fundamental quantum-computing resource to speedup essentially a quantum computation.

<sup>3</sup>The definition of the solution state of a search problem in this paper is consistent with those in a conventional quantum search algorithm [3, 13]. Given a Boolean function  $f : \{0, 1, \dots, N-1\} \rightarrow \{0, 1\}$  there exists a unique element  $x_0 \in \{0, 1, \dots, N-1\}$  such that  $f(x) = 1$  if  $x = x_0$  and  $f(x) = 0$  if  $x \neq x_0$ . Then the solution state is defined as the quantum state corresponding to the element  $x_0$ . Sometimes the solution state also is called the marked state, etc..

this second step. It is devoted to investigating the principle and mechanism of how a UNIDYSLOCK process and a QUANSDAM process work in a quantum system. In the future paper [36] the author will report in detail an exponential QUANSDAM process in a single-atom system.

The *HSSS* quantum search process is essentially different from the conventional quantum search algorithm (i.e., the Grover's quantum search algorithm [3]). Though they obey the same mathematical-logical principle that the unstructured search problem obeys, in algorithm they essentially differ from one another mainly in the two aspects as follows. The first aspect is that what the conventional quantum search algorithm searches for is the solution state of the search problem as a whole, while what the *HSSS* quantum search process searches for is any single component state of the solution state. This difference really reflects that there must be the search-space dynamical reduction for the *HSSS* quantum search process. The second aspect is that as far as their basic building blocks, i.e., the black-box functional operations are concerned, they obey essentially different mathematical-logical principles of black-box functional operation, respectively. The oracle operation (i.e., the black-box functional operation) of a conventional quantum search algorithm [3, 13] is a quantum parallel functional operation in the Hilbert space of the quantum system. In contrast, the black-box functional operation of the *HSSS* quantum search process works not only in the Hilbert space of the quantum system but also in the mathematical Hilbert space (i.e., the math Hilbert space) of the search problem. In the Hilbert space of the quantum system it is considered as a single black-box functional operation and hence does not have the mathematical-logical meaning of quantum parallel operation and in the math Hilbert space it may be thought of as a mathematical-parallel black-box functional operation that is essentially different from a conventional quantum parallel functional operation [11, 10, 9]. This difference indicates that there is the quantum-state-difference amplification for the *HSSS* quantum search process, which is essentially different from the amplitude amplification of a conventional quantum search algorithm.

The *HSSS* quantum search process does not employ the quantum parallel principle of the conventional quantum computation theory [11a] (the essential aspect of the principle is the quantum-state entanglement and nonlocal effect) to achieve its quantum-searching speedup. Instead, its quantum-searching speedup is original from the interaction between the quantum physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) and the mathematical-logical principle that the unstructured search problem obeys [1]. Because the *HSSS* quantum search process has its own quantum-computing speedup mechanism essentially different from the one of a conventional quantum search algorithm, in which it eliminates the unstructured search space and obeys the mathematical-logical principle of black-box functional operation that is essentially different from the one obeyed by a conventional quantum search algorithm (See the preceding paragraph), the well-known square speedup limit [4a] on a conventional quantum search algorithm could not be applied to it. Therefore, it could have a super-square or even an exponential quantum-searching speedup.

It is well known that a conventional quantum search algorithm to solve an unstructured search problem can achieve a square search speedup [3], while its classical counterpart can achieve at best a linear search speedup; and moreover, this square speedup is the maximum one in the worst case [4] (i.e., the square speedup limit). It has been believed extensively that hardness to solve an unstructured search problem is closely related to the unstructured symmetric property of search space of the search problem both in classical computation and in quantum computation. Intuitively one may say that due to its unstructured property the unstructured search space could prohibit any conventional quantum and classical search algorithms from achieving an exponential searching speedup. A reasonable explanation for this phenomenon is simply given below.

A quantum search process (or algorithm) obeys not only the unitary quantum dynamics but also the mathematical-logical principle such as the symmetric structure of unstructured search space of the search problem [5, 6, 1]. There is no exception for a conventional quantum search algorithm. When the conventional quantum search algorithm is performed in a quantum system, the unstructured symmetric property of search space may make a severe constraint on the time evolution process of the quantum system. As a consequence of this constraint, the quantum system could be forced to run into the regime of classical physics, here or there and early or later, during the quantum search process, no matter that it is a single quantum system such as a single-atom system or a composite quantum system such as an  $n$ -qubit quantum system, and no matter which quantum state it takes at the initial time (its quantum state at any later time is determined completely by the initial state according to the unitary quantum dynamics). This consequence implies that a conventional quantum search algorithm is semiclassical [1]. Ultimately it results in that the quantum-searching speedup of the algorithm is greatly suppressed and the algorithm becomes inefficient in the worst case.

If the unstructured search space is not eliminated and the classical effect hidden behind the unstructured search space is not purged, any unstructured quantum search algorithm is not able to achieve an exponential speedup. Therefore, the very first task for any possible efficient unstructured quantum search process (or algorithm) is naturally to eliminate this unstructured search space.

The search-space dynamical reduction, i.e., the first step of the *HSSS* quantum search process is just used to eliminate this unstructured search space. Obviously, it is not a usual computational complexity reduction. The latter means that one computational problem may be solved by reducing it to another which may be solved conveniently on quantum computer. In concept it has nothing to do with the fundamental quantum-computing resource.

It is not free to eliminate an unstructured search space. First of all, the fundamental quantum-computing resource is necessary to realize a search-space dynamical reduction. Then a search-space dynamical reduction even makes it harder to extract the information of the component states of the solution state of a search problem from the amplitude-amplification point of view of a conventional quantum search algorithm. It is known [3] that the initial amplitude

(i.e., the minimum amplitude) of the solution state is proportional to  $1/\sqrt{N}$  in a conventional quantum search algorithm with search-space dimension  $N = 2^n$ . However, in the *HSSS* quantum search process, after a search-space dynamical reduction (here the unitary state-transformation description is used) the quantum state of the polynomially-small reduction subspace (with dimension two) that carries the solution information of the component state has an amplitude proportional to  $1/N$  [2]. This amplitude is exponentially small. Moreover, it is even exponentially smaller than its counterpart ( $1/\sqrt{N}$ ) in a conventional quantum search algorithm. It seems that a search-space dynamical reduction makes thing worse. This, of course, concludes from the amplitude-amplification viewpoint of a conventional quantum search algorithm.

Here an exponentially-small amplitude for the final quantum state in a search-space dynamical reduction that carries the solution information means an exponentially-small quantum-state difference between the possible quantum states created at the end of the search-space dynamical reduction. The latter, i.e., the quantum-state difference, is closely related to the characteristic physical quantity, i.e., the quantum-state-difference varying rate, of a UNIDYSLOCK process and a QUANSDAM process in the *HSSS* quantum search process [1].

After the search-space dynamical reduction, the *HSSS* quantum search process performs the inverse of a UNIDYSLOCK process, i.e., the QUANSDAM process to amplify the quantum-state difference [1], so that the solution information can be obtained for the search problem. This is just the second step of the *HSSS* quantum search process. Since the unstructured search space is eliminated, there could not exist such a severe constraint as the square speedup limit [4] on the quantum-state-difference (*QSD*) amplification ability (i.e., the *QSD* varying speed) of the QUANSDAM process in the *HSSS* quantum search process. Because the *QSD* amplification ability is original from the interaction between the quantum physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) and the mathematical-logical principle of the search problem, there could be a QUANSDAM process that has a super-square or even an exponential *QSD* amplification ability. If the *QSD* amplification ability is large enough, then it can compensate not only the amplitude loss due to the search-space dynamical reduction (See the paragraphs above) but also it enables the *HSSS* quantum search process as a whole to achieve a super-square or even an exponential quantum-searching speedup. Therefore, the *QSD* amplification ability of a QUANSDAM process is a key factor to measure whether the *HSSS* quantum search process is successful or not.

Originally the concept of unitary dynamical state-locking (UNIDYSLOCK) process is connected with the search-space dynamical reduction in the *HSSS* quantum search process [5, 6]. It was made an attempt in Ref. [6] to use the reversible and unitary halting protocol to realize the search-space dynamical reduction,<sup>4</sup> while it was proposed to use a UNIDYSLOCK process [6, 7] to

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<sup>4</sup>So far there still lacks a rigorous mathematical proof to show that a reversible and unitary halting protocol can realize a search-space dynamical reduction. Obviously, any conventional

realize the reversible and unitary halting protocol.

Initially a UNIDYSLOCK process was constructed in Ref. [7] with the help of a single-atom system. It was used to convert unitarily the solution state (or its component state) of the search problem into the final state that is sufficiently close to the desired (known) state with the help of the reversible and unitary halting protocol. It was thought at that time that the final state might be prepared more easily. Then the inverse of the UNIDYSLOCK process could change unitarily the final state back to the solution state (or its component state) with the help of the inverse of the reversible and unitary halting protocol. Therefore, a UNIDYSLOCK process and its inverse process, i.e., the QUANS DAM process could be used to solve the unstructured search problem [6, 7]. It is known that the solution state can be any state of the unstructured search space of the search problem. This implies that a UNIDYSLOCK process could convert unitarily many different (or orthogonal) quantum states into their corresponding non-orthogonal final states at the same time, respectively, which are sufficiently close to each other and to the desired state. Unfortunately, the research into a UNIDYSLOCK process in past years found that in pure quantum mechanics a UNIDYSLOCK process does not make sense and cannot have an ability to convert unitarily many orthogonal quantum states into their corresponding non-orthogonal states at the same time.

Actually the research also finds that in classical reversible computation [8] and any conventional quantum computation [9, 10, 11] any computational process which is reversible (or unitary) can not transform many orthogonal quantum states to their corresponding non-orthogonal states at the same time. This raises question as to whether a UNIDYSLOCK process does exist really in quantum computation.

In nature only a non-equilibrium irreversible process is able to change many orthogonal quantum states to their corresponding non-orthogonal states at the same time. It is well known that in reality there is a huge conflict between a non-equilibrium irreversible process and a reversible process or a quantum-mechanical unitary dynamical process. The latter two processes are not able to change the quantum-state difference. The largest puzzle for a UNIDYSLOCK process then is that it is a unitary quantum dynamical process, but it wants to change simultaneously many orthogonal quantum states to their corresponding non-orthogonal states.

A UNIDYSLOCK process [6, 7] is not only a unitary quantum dynamical process but it also obeys the mathematical-logical principle of the computational problem to be solved (i.e., an unstructured search problem here). Therefore, a complete description for a UNIDYSLOCK process must consider both the aspect of the unitary quantum dynamics and the aspect of the mathematical-logical principle [1]. This is quite like that a microscopic particle is described by the wave-particle dualism in quantum mechanics, that is, a microscopic particle can be described completely only when one considers both the particle aspect and

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halting operation (or protocol) that is involved in quantum measurement is not suited to realize a search-space dynamical reduction.

the wave aspect of the same particle. Both the aspects of the unitary quantum dynamics and the mathematical-logical principle together and especially their interaction for a UNIDYSLOCK process are really key to understanding completely what a UNIDYSLOCK process means exactly and how it works and to answering all its puzzles.

On the basis of the quantum-computing speedup theory [1] this paper is devoted to revealing the principle and mechanism of how a UNIDYSLOCK process and a QUANSDAM process work in a quantum system. A UNIDYSLOCK (or QUANSDAM) process is the only nontrivial and unitary quantum-physical process that can change the quantum-state difference between a pair of quantum states. It is essentially different from a usual non-equilibrium irreversible process which could contain the usual quantum measurement. It is the characteristic quantum computational process (or sub-process) of the quantum-computing speedup theory. Whether or not its contribution to a quantum-computing speedup is dominating can distinguish the quantum-computing speedup theory from any conventional quantum computational theory. In the paper the research is also carried out on the quantum-computing speedup mechanism of the *HSSS* quantum search process. The *HSSS* quantum search process works in both the Hilbert space of a quantum system and the math Hilbert space of the search problem. In contrast, a conventional quantum search algorithm [3, 13] works only in the Hilbert space of the quantum system. Here the math Hilbert space of a computational problem is a fundamental concept of the quantum-computing speedup theory. The concept does not exist in conventional quantum computation. Due to the math Hilbert space a mathematical-parallel operation is the characteristic manner to realize a reversible functional operation in the quantum-computing speedup theory. In contrast, a quantum parallel functional operation is the characteristic manner in the conventional quantum computational theory based on the quantum parallel principle [11a]. The fundamental interaction between the quantum physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) and the mathematical-logical principle that a computational problem obeys is essentially responsible for a quantum-computing speedup in the quantum-computing speedup theory. This is a fundamental principle of the theory. It is available not only for a UNIDYSLOCK (or QUANSDAM) process but also for the whole *HSSS* quantum search process.

There are six sections in the paper. The section 2 describes the relation between the search-space dynamical reduction and the information-carrying unitary operator. The section 3 is devoted to investigating the principle and mechanism of how a UNIDYSLOCK process and a QUANSDAM process work in a quantum system. The section 4 describes how to prepare the information-carrying unitary propagators of a quantum system and especially a single-atom system. The section 5 describes the mechanism of the *HSSS* quantum search process and the essential difference between the *HSSS* quantum search process and the conventional quantum search algorithm. The final section is devoted to discussing the essential difference between the quantum-computing speedup theory and any conventional quantum computational theory.



## 2. The efficient search-space dynamical reduction and the information-carrying unitary operators

The search-space dynamical reduction was proposed in Ref. [5a] for the first time. It was realized efficiently in Ref. [2]. Its purpose is to realize that the exponentially large unstructured search space of a search problem is reduced dynamically to a polynomially small subspace. The unstructured search space of a search problem is mathematical. It is different from the usual Hilbert space of a quantum system used to solve the search problem. From the mathematical-logical viewpoint a direct reduction of the unstructured search space is not allowed [1]. However, the quantum-computing speedup theory [1] does allow one to reduce indirectly and dynamically the unstructured search space through the unitary oracle selective diagonal operator in the frame of unitary quantum dynamics. This reduction needs to use the fundamental quantum-computing resource and is shaped by the Hilbert-space symmetric structure of the quantum system. A search-space dynamical reduction shows that in the quantum-computing speedup mechanism [1] one needs to consider explicitly not only the quantum-physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) that governs the quantum system used to solve the search problem but also the mathematical-logical principle (such as the unstructured symmetric property of the search space) obeyed by the search problem and especially the interaction between the quantum-physical laws and the mathematical-logical principle. This is essentially different from the conventional quantum computational theory [11].

There are two different Hilbert spaces that need to be considered explicitly in the quantum-computing speedup theory [1]. One of which is the Hilbert space of a quantum system that is used to solve a computational problem. It is quantum physical. Its symmetric structure is the fundamental quantum-computing resource. Another is of the computational problem to be solved in the quantum system. It is a linear complex vector space that accommodates the mathematically operational space of function of the computational problem. It is mathematical and does not have any physical meaning. For convenience, hereafter this linear complex vector space is called the mathematical (or math) Hilbert space of the computational problem. The math Hilbert space of a computational problem is a fundamental concept in the quantum-computing speedup theory. There does not exist independently the concept of the math Hilbert space in any conventional quantum computation. The very reason why the quantum-computing speedup theory needs to consider explicitly both the Hilbert spaces is that in the theory an essential quantum-computing speedup is original from the interaction between the quantum-physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) obeyed by a quantum system and the mathematical-logical principle obeyed by a computational problem to be solved in the quantum system.

The quantum-computing speedup theory [1] then pointed out that the math Hilbert space (e.g., the unstructured search space) may be considered as the

Hilbert space of the quantum system in conventional quantum computation. This is the special case. Equivalently it may be considered that the Hilbert space of a quantum system also acts as the math Hilbert space in conventional quantum computation, and then both the Hilbert space and the math Hilbert space are treated together as one space. Clearly there is not a separate math Hilbert space in conventional quantum computation. There is only the Hilbert space of the quantum system. There is no consideration of the interaction between the two Hilbert spaces. These results show that there is no exploitation of the Hilbert-space symmetrical structure of a quantum system to speed up a quantum computation in conventional quantum computation [9, 10, 11].

In contrast, in the quantum-computing speedup theory one has to treat explicitly and separately the Hilbert space of a quantum system and the math Hilbert space of a computational problem, because the *HSSS* quantum search process needs first to perform the search-space dynamical reduction [5, 6, 1, 2] for the unstructured search space which is just the math Hilbert space of an unstructured search problem. Moreover, one has to set up suitably the interaction between the two Hilbert spaces so that the Hilbert-space symmetric structure of the quantum system (i.e., the fundamental quantum-computing resource) can be harnessed to speed up a quantum computation.

The difference for how to treat these two different Hilbert spaces mentioned above between the quantum-computing speedup theory [1] and the conventional quantum computational theory based on the quantum parallel principle [11a] really reflects two different understandings for the origin of quantum-computing speedup. For the quantum-computing speedup theory an essential quantum-computing speedup is original from the interaction between the quantum-physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) and the mathematical-logical principle that a computational problem obeys, while for the conventional quantum computational theory an essential quantum-computing speedup is original from the quantum-state effects which may include the quantum-state superposition, coherence interference, entanglement and non-local effect, correlation and so on. (Among these quantum-state effects the essential one is the quantum entanglement and non-local effect.) The difficulties and limitations for the conventional quantum computational theory [11] have been exposed in detail in Ref. [1].

A typical example is given below to help understand both the Hilbert space of a quantum system and the math Hilbert space of a computational problem. Suppose that the search space of an unstructured search problem is given by  $\{0, 1, \dots, N-1\}$  in the (mathematical) number representation or by  $\{|0\rangle, |1\rangle, \dots, |N-1\rangle\}$  in the vector (or state) representation with the usual computational basis vectors  $\{|k\rangle\}$  ( $0 \leq k < N$ ). The latter may be called the math Hilbert space of the search problem. It is a linear complex vector space with dimensional size  $N$ . It is mathematical and unstructured. It does not have any physical meaning, although here one uses the vector (or state) set  $\{|0\rangle, |1\rangle, \dots, |N-1\rangle\}$  to stand for it. It is different from the usual Hilbert space of a quantum system. The latter is quantum physical. As usual, here the Hilbert space of a quantum system with dimensional size  $N$  may be given by

$\{|0\rangle, |1\rangle, \dots, |N-1\rangle\}$ , where  $\{|k\rangle\}$  ( $0 \leq k < N$ ) are the usual computational basis states of the quantum system.

As shown in the section 5 below, both the Hilbert space of a quantum system and the math Hilbert space of an unstructured search problem are treated in different manners by a conventional quantum search algorithm [3, 13] and the *HSSS* quantum search process, respectively. In a conventional quantum search algorithm an  $n$ -qubit quantum system is usually employed to solve an unstructured search problem with  $2^n$ -dimensional search space [3] and its Hilbert space  $\{|0\rangle, |1\rangle, \dots, |N-1\rangle\}$  with  $N = 2^n$  just acts as the unstructured search space  $\{|0\rangle, |1\rangle, \dots, |N-1\rangle\}$  of the search problem<sup>5</sup>. Therefore, both the Hilbert space of the quantum system and the math Hilbert space of the search problem are the same one and coincident completely with one another. They are treated together as one space [3, 13] and any one of the Hilbert space and the math Hilbert space is no longer considered separately in the algorithm. Therefore, one may say that a conventional quantum search algorithm works in the Hilbert space of a quantum system only. However, in the *HSSS* quantum search process both the Hilbert spaces have to be treated separately [1, 6, 5], although they are the same one. This means that the *HSSS* quantum search process works in two different Hilbert spaces, one being of the quantum system and another of the search problem.

The interaction between the two Hilbert spaces may be best understood through the unitary oracle selective diagonal operator which is the basic building block of the *HSSS* quantum search process. Below this research topic is described simply. A more detailed description for the research topic should refer Refs. [5, 6, 1, 2] and the section 5 below. The unitary oracle selective diagonal (or phase-shift) operator  $C_S(\theta)$  may be written as [5, 2]

$$C_S(\theta) = \exp(-i\theta D_S). \quad (2.1)$$

Here  $D_S$  is called the oracle (or black-box) selective diagonal operator. It is defined by (2.3) below or equivalently by  $D_S = |S\rangle\langle S|$  with the usual computational basis state  $|S\rangle$  given by (2.4) below. It is required here that the state  $|S\rangle$  take any possible (or candidate) solution state of the unstructured search space of the search problem. The reason why the state  $|S\rangle$  represents any candidate solution state is given in the section 5. Sometimes  $C_S(\theta)$  is called the unitary oracle selective phase-shift operator [5a]. The unitary oracle selective diagonal operator  $C_S(\theta)$  in the *HSSS* quantum search process is functionally equivalent to the usual oracle operation  $U_o(\theta)$  (i.e., the basic building block) in a conventional quantum search algorithm. The latter is defined by [3, 13]

$$U_o(\theta) : |x\rangle \rightarrow \begin{cases} \exp(-i\theta) |x\rangle & \text{if } f(x) = 1 \\ |x\rangle & \text{if } f(x) = 0 \end{cases} \quad (2.2)$$

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<sup>5</sup>Obviously in addition to this pivotal  $n$ -qubit quantum system and the unstructured search space there could be other auxiliary quantum systems and mathematically operational spaces of function in the quantum search algorithm. They are mainly used to help realize the quantum search algorithm.

where the function  $f(x)$  is defined by  $f(x) = 1$  if  $x = x_0$  and  $f(x) = 0$  if  $x \neq x_0$  and  $|x_0\rangle$  denotes the unique solution state of the search problem (See also Footnote 3 in this paper). Therefore, here  $C_S(\theta)$  and  $U_o(\theta)$  also may be called the black-box functional operations of the *HSSS* quantum search process and a conventional quantum search algorithm, respectively. Note that the original oracle operation  $U_o(\pi)$  [3] is basic, (two  $U_o(\pi)$  plus one auxiliary qubit are sufficient to generate any  $U_o(\theta)$ ), and the mathematical-logical meaning of  $C_S(\theta)$  is determined completely by the Boolean functional operation (or equivalently by the mathematical-logical principle of the search problem), as shown in the section 5. The mathematical-logical meaning for  $C_S(\theta)$  and  $U_o(\theta)$  then has nothing to do with the pure parameter  $\theta$ .

As shown in the section 5, both  $C_S(\theta)$  and  $U_o(\theta)$  can represent faithfully the Boolean function-operational sequence *BFSEQ* in (5.3) or (5.4), indicating that they obey the same mathematical-logical principle of the search problem. Therefore, each one of  $C_S(\theta)$  and  $U_o(\theta)$  can be used as the basic building block or the black-box functional operation to construct a black-box quantum search algorithm to solve the search problem. However, in quantum computation the unitary oracle selective diagonal operator  $C_S(\theta)$  is essentially different from the usual oracle operation  $U_o(\theta)$ . It is well known that the usual oracle operation  $U_o(\theta)$  [3, 13] is a quantum parallel operation that works in the Hilbert space of a quantum system and it is irreducible in the mathematical-logical meaning of unstructured search. In contrast, as shown in the section 5, the unitary oracle selective diagonal operator  $C_S(\theta)$  works not only in the Hilbert space of a quantum system but also in the math Hilbert space of the search problem. It is a single black-box functional operation in the Hilbert space of the quantum system and does not have the mathematical-logical meaning of quantum parallel operation, and it may be considered as a mathematical-parallel black-box functional operation in the math Hilbert space. Moreover, it is allowed to be reducible in the mathematical-logical meaning of unstructured search. Note that a mathematical-parallel functional operation is essentially different from a quantum-parallel functional operation (See the section 5). These substantial mathematical-logical differences between  $C_S(\theta)$  and  $U_o(\theta)$  result in that the *HSSS* quantum search process is essentially different from a conventional quantum search algorithm, although they solve the same unstructured search problem.

According to the quantum-computing speedup theory there are  $N$  candidate solution states if the unstructured search space of the search problem is  $N$ -dimensional. Here one may employ the  $N$  usual computational basis vectors  $\{|S\rangle\}$  ( $0 \leq S < N$ ) of the unstructured search space  $\{|0\rangle, |1\rangle, \dots, |N-1\rangle\}$  to represent these  $N$  candidate solution states, respectively. One also may use the  $N$  usual computational basis states  $\{|S\rangle\}$  ( $0 \leq S < N$ ) of the Hilbert space  $\{|0\rangle, |1\rangle, \dots, |N-1\rangle\}$  of a quantum system to represent these  $N$  candidate solution states, respectively. In either case all these  $N$  candidate solution states are clearly orthogonal to each other. The theory demands to set up suitably the interaction between the math Hilbert space (i.e., the unstructured search space) and the Hilbert space of the quantum system used to solve the search problem.

It will be shown below that the interaction may be set up through the oracle selective diagonal operator  $D_S$  or the unitary oracle selective diagonal operator  $C_S(\theta)$ . First consider the conventional case that the unstructured search space is  $2^n$ -dimensional, i.e.,  $N = 2^n$ . Then in usual quantum computation the search problem usually may be solved in an  $n$ -qubit quantum system [3, 13]. On the one hand, the  $n$ -qubit quantum system provides the fundamental quantum-computing resource. For convenience, here it is set to an  $n$ -qubit spin (or pseudospin) system, so that the oracle selective diagonal operator  $D_S$  may be characterized by the  $z$ -component spin-1/2 (or pseudospin-1/2) operators  $\{I_{1z}, I_{2z}, \dots, I_{nz}\}$  of the  $n$ -qubit spin system. On the other hand, every one of these  $N$  candidate solution states  $\{|S\rangle\}$  ( $0 \leq S < N$ ) can be represented uniquely by a unit-number vector (or the double-value logical number vector)  $\{a_1^s, a_2^s, \dots, a_n^s\}$  with the unit number (or the double-value logical number)  $a_k^s = \pm 1$  ( $1 \leq k \leq n$ ). Note that these  $N$  candidate solution states, which are the  $N$  usual computational basis vectors, form an orthonormal basis set of the unstructured search space. Therefore, the unit-number vector  $\{a_1^s, a_2^s, \dots, a_n^s\}$  can represent faithfully the unstructured search space  $\{|0\rangle, |1\rangle, \dots, |N-1\rangle\}$ . Now in the unit-number representation it can prove that the oracle selective diagonal operator  $D_S$ , which also is given by  $D_S = |S\rangle\langle S|$ , may be expressed as [5, 2]

$$D_S = (\frac{1}{2}E_1 + a_1^s I_{1z}) \otimes (\frac{1}{2}E_2 + a_2^s I_{2z}) \otimes \dots \otimes (\frac{1}{2}E_n + a_n^s I_{nz}). \quad (2.3)$$

Here the unit-number vector  $\{a_1^s, a_2^s, \dots, a_n^s\}$  also may be used to stand for any candidate solution state  $|S\rangle = |s_1\rangle|s_2\rangle\dots|s_n\rangle$  in the Hilbert space of the  $n$ -qubit spin system, and the latter may be expressed as

$$|S\rangle = (\frac{1}{2}|T_1\rangle + a_1^s|S_1\rangle) \otimes (\frac{1}{2}|T_2\rangle + a_2^s|S_2\rangle) \otimes \dots \otimes (\frac{1}{2}|T_n\rangle + a_n^s|S_n\rangle) \quad (2.4)$$

where  $|T_k\rangle = |0_k\rangle + |1_k\rangle$  and  $|S_k\rangle = \frac{1}{2}(|0_k\rangle - |1_k\rangle)$ , and  $|s_k\rangle = |0_k\rangle$  or  $|1_k\rangle$  is the  $k$ -th component state of the candidate solution state; and the operator  $E_k$  and  $I_{kz}$  are the unit operator and the  $z$ -component spin operator of the  $k$ -th spin-1/2 of the  $n$ -qubit spin system, respectively. The eigenvalue equation for the spin operator  $I_{kz}$ <sup>6</sup> is  $I_{kz}|0_k\rangle = \frac{1}{2}|0_k\rangle$  and  $I_{kz}|1_k\rangle = -\frac{1}{2}|1_k\rangle$ . It is easy to find that the number (or index)  $S$  that corresponds to the candidate solution state  $|S\rangle$  is given by

$$S = \frac{1}{2}(1 - a_1^s) \times 2^{n-1} + \frac{1}{2}(1 - a_2^s) \times 2^{n-2} + \dots + \frac{1}{2}(1 - a_n^s) \times 2^0. \quad (2.5)$$

These formulae (2.1), (2.3), (2.4), and (2.5) set up the one-to-one corresponding relations between the unitary oracle selective diagonal operator  $C_S(\theta)$ , the oracle selective diagonal operator  $D_S$ , the candidate solution state  $|S\rangle$ , and the

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<sup>6</sup>The spin operator  $I_\mu$  of a single spin-1/2 is related to the Pauli operator  $\sigma_\mu$  by  $I_\mu = \frac{1}{2}\sigma_\mu$  for  $\mu = x, y, z$

(index) number  $S$ . Here the unit-number vector  $\{a_1^s, a_2^s, \dots, a_n^s\}$  plays the central role in setting up these corresponding relations. Similarly, one also may set up these one-to-one corresponding relations in the math Hilbert space and here one may consider that the ‘ $n$ -qubit spin system’ associated with the math Hilbert space is completely coincident with the  $n$ -qubit spin system. Therefore, the expression of (2.4) also may be used to represent the candidate solution state  $|S\rangle$  of the math Hilbert space as long as the ‘ $n$ -qubit spin system’ is considered as the  $n$ -qubit spin system.

A complete understanding for the unit number  $a_m^s$  ( $1 \leq m \leq n$ ) must be from the two aspects; One aspect is the logical number ( $a_m^s = \pm 1$ ) and another the information of the component state ( $|s_m\rangle$ ) of the solution state. Lacking either aspect will result in a non-complete understanding. In particular, one must not consider that the unit number  $a_m^s$  is merely a pure parameter. Otherwise there is no mathematical-logical meaning for  $C_S(\theta)$ .

In quantum computation this type of unitary selective diagonal operators  $\{C_k(\theta)\}$  [5, 25b, 12] could have an extensive application in future to solving many hard problems (See, for example, Refs. [3, 13, 25, 26]). Such an application was reported in Ref. [5a] for the first time, where the unstructured search problem is a research topic, and the search-space dynamical reduction is realized theoretically in the unit-number representation  $\{a_1^s, a_2^s, \dots, a_n^s\}$  in which  $C_k(\theta)$  ( $k = S$ ) is the so-called unitary oracle selective diagonal operator  $C_S(\theta)$  mentioned above. The mathematical-logical meaning for the unitary oracle selective diagonal operator  $C_S(\theta)$  in the *HSSS* quantum search process is described in the section 5. The key point in the application is that the (mathematical) number  $k$ , the computational basis state  $|k\rangle$ , and the unitary selective diagonal operator  $C_k(\theta) = \exp(-i\theta D_k)$  or the selective diagonal operator  $D_k = |k\rangle\langle k|$  must be one-to-one correspondent to each other; and moreover, the basis state  $|k\rangle$  and the selective diagonal operator  $D_k = |k\rangle\langle k|$  or  $C_k(\theta)$  must reflect the symmetric structure of the tensor-product Hilbert space of the quantum system used to solve a computational problem.

The unitary oracle selective diagonal operator acts as a bridge to connect the mathematical-logical principle obeyed by the search problem and the quantum physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) obeyed by the quantum system. Now the oracle selective diagonal operator of (2.3) can tell ones that there exists the interaction between the Hilbert space of the  $n$ -qubit spin system and the math Hilbert space of the search problem. Note that in the quantum-computing speedup theory the math Hilbert space is not coincident with the Hilbert space completely. On the one hand, as shown above, the unit-number vector  $\{a_1^s, a_2^s, \dots, a_n^s\}$  in the oracle selective diagonal operator  $D_S$  is able to represent the unstructured search space of the search problem. On the other hand, all these  $n$   $z$ -component spin operators  $\{I_{kz}\}$  in the oracle selective diagonal operator act on only the  $n$ -qubit spin system. Then by expanding (2.3) one can find that the oracle selective diagonal operator  $D_S$  really contains a variety of coupled terms (e.g.,  $\{a_k^s I_{kz}\}$ ,  $\{a_k^s a_l^s I_{kz} I_{lz}\}$ , etc.) of both the Hilbert space of the  $n$ -qubit spin system and the unstructured search space, indicating that there exists the interaction

between the two Hilbert spaces.

A slightly more complex instance than the above one may be helpful for understanding more deeply the interaction between both the Hilbert spaces. Suppose that an  $n$ -qutrit spin (or pseudospin) system that consists of  $n$  spins (or pseudospins) with spin quantum number  $I = 1$  is used to solve the unstructured search problem. It provides the fundamental quantum-computing resource. A spin with  $I = 1$  has three basis states in its Hilbert space, which may be set to the three eigenstates  $|+1\rangle$ ,  $|0\rangle$ , and  $|-1\rangle$  of the  $z$ -component spin operator  $I_z$  with eigenvalues  $+1$ ,  $0$ ,  $-1$ , respectively, here the eigenvalue equation for the operator  $I_z$  is given by  $I_z|m\rangle = m|m\rangle$  with  $m = +1, 0, -1$ . In quantum computation these three basis states  $|+1\rangle$ ,  $|0\rangle$ , and  $|-1\rangle$  may represent the three computational basis states  $|0\rangle$ ,  $|1\rangle$ , and  $|2\rangle$  of a single-qutrit system, respectively. Then for the  $n$ -qutrit spin system it can turn out that the oracle selective diagonal operator  $D_S = |S\rangle\langle S|$  may be expressed as

$$D_S = \bigotimes_{k=1}^n \left( (1 - |a_k^s|) E_k + \frac{1}{2} a_k^s I_{kz} + \left( -1 + \frac{3}{2} |a_k^s| \right) I_{kz}^2 \right) \quad (2.6)$$

where the triple-value logical number  $a_k^s = +1, 0, -1$ ; and the unit operator  $E_k$  and the spin operator  $I_{kz}$  are the diagonal operators of the  $k$ -th spin-1 (or pseudospin-1), i.e.,  $E_k = \text{Diag}(1, 1, 1)$ ,  $I_{kz} = \text{Diag}(1, 0, -1)$ , and  $I_{kz}^2 = \text{Diag}(1, 0, 1)$ . Correspondingly any candidate solution state  $|S\rangle$  (or  $|S\rangle$ ) of the search problem is given by

$$|S\rangle = \bigotimes_{k=1}^n \left( (1 - |a_k^s|) |T_{+1}^k\rangle + \frac{1}{2} a_k^s |T_0^k\rangle + \left( -1 + \frac{3}{2} |a_k^s| \right) |T_{-1}^k\rangle \right) \quad (2.7)$$

where  $|T_{+1}^k\rangle = |+1_k\rangle + |0_k\rangle + |-1_k\rangle$ ,  $|T_0^k\rangle = |+1_k\rangle - |-1_k\rangle$ , and  $|T_{-1}^k\rangle = |+1_k\rangle + |-1_k\rangle$ . Here one employs the triple-value logical number vector  $\{a_1^s, a_2^s, \dots, a_n^s\}$  with the triple-value logical number  $a_k^s = +1, 0, -1$  ( $1 \leq k \leq n$ ) to express (or construct) the oracle selective diagonal operator  $D_S$ . Every candidate solution state  $|S\rangle$  of the search problem is represented one-to-one by a triple-value logical number vector  $\{a_1^s, a_2^s, \dots, a_n^s\}$  in an  $n$ -qutrit spin system. Furthermore, readers may refer Refs. [6, 12] for a general construction of  $D_S$  and  $|S\rangle$  in a more complex quantum system. Now by using the triple-value logical number  $a_k^s = +1, 0, -1$  the  $k$ -th component state  $|s_k\rangle$  of the candidate solution state  $|S\rangle = |s_1\rangle|s_2\rangle\dots|s_n\rangle$  may be written as  $|s_k\rangle = (1 - |a_k^s|) |T_{+1}^k\rangle + \frac{1}{2} a_k^s |T_0^k\rangle + \left( -1 + \frac{3}{2} |a_k^s| \right) |T_{-1}^k\rangle = |+1_k\rangle, |0_k\rangle$ , and  $|-1_k\rangle$ , respectively. It is also easy to find that the (index) number  $S$  corresponding to the state  $|S\rangle$  is given by

$$S = (1 - a_1^s) \times 3^{n-1} + (1 - a_2^s) \times 3^{n-2} + \dots + (1 - a_n^s) \times 3^0. \quad (2.8)$$

Thus, the formulae (2.1), (2.6), (2.7), and (2.8) also set up the one-to-one corresponding relations between  $C_S(\theta)$ ,  $D_S$ ,  $|S\rangle$ , and  $S$  through the triple-value logical number vector  $\{a_1^s, a_2^s, \dots, a_n^s\}$ . Moreover, the oracle selective diagonal

operator of (2.6) and the candidate solution state of (2.7) indeed reflect the symmetric structure of the tensor-product Hilbert space of the  $n$ -qutrit spin system.

The triple-value logical number vector  $\{a_1^s, a_2^s, \dots, a_n^s\}$  still stands for any candidate solution state  $|S\rangle$  (or  $|S\rangle$ ) and it is able to represent the unstructured search space  $\{|0\rangle, |1\rangle, \dots, |N-1\rangle\}$  with  $N = 3^n$  of the search problem, because all these candidate solution states form an orthonormal basis set of the unstructured search space. On the other hand, in the oracle selective diagonal operator of (2.6) these  $z$ -component spin operators  $\{I_{kz}\}$  and  $\{I_{kz}^2\}$  act on only the  $n$ -qutrit spin system. By expanding (2.6) one can find that the oracle selective diagonal operator consists of a variety of interaction terms of both the Hilbert space of the  $n$ -qutrit spin system and the math Hilbert space of the search problem. Thus, there exists the interaction between the two Hilbert spaces.

As can be seen in (2.3) and (2.6), the oracle selective diagonal operator of (2.6) associated with the  $n$ -qutrit spin system is quite different from that one of (2.3) with the  $n$ -qubit spin system. The interacting terms between the two Hilbert spaces in the oracle selective diagonal operator with the  $n$ -qutrit spin system are much more complex than those in the oracle selective diagonal operator with the  $n$ -qubit spin system. Both the  $n$ -qubit and  $n$ -qutrit spin systems have the tensor-product Hilbert spaces with different symmetric structures, respectively. As shown in (2.3) and (2.6), the tensor-product symmetric structure appearing in one oracle selective diagonal operator is determined completely by the Hilbert space of the quantum system associated with the oracle selective diagonal operator. These indicate that *the symmetric structure of tensor-product Hilbert space of a quantum system shapes the interaction between the Hilbert space of the quantum system and the math Hilbert space of the unstructured search problem to be solved.*

Below an  $n$ -qubit spin (or pseudospin) system will be studied mainly, while an  $n$ -qutrit spin (or pseudospin) system will not be further discussed unless stated otherwise.

The search-space dynamical reduction was first constructed in Ref. [5a] in the unit-number representation  $\{a_1^s, a_2^s, \dots, a_n^s\}$  in an  $n$ -qubit spin quantum system. The interaction between the Hilbert space of the quantum system and the math Hilbert space of the search problem is necessary to realize the search-space dynamical reduction. As shown above, it is shaped by the symmetric structure of the tensor-product Hilbert space of the quantum system. This is the reason why the fundamental quantum-computing resource is necessary to realize a search-space dynamical reduction. An efficient search-space dynamical reduction is already realized in an  $n$ -qubit spin (or pseudospin) system theoretically [2]. Here the unit-number representation  $\{a_1^s, a_2^s, \dots, a_n^s\}$  of the  $n$ -qubit spin system is fundamentally important to realize the efficient search-space dynamical reduction. It is used to express the oracle selective diagonal operator  $D_S$  and the unitary oracle selective diagonal operator  $C_S(\theta)$  which are given by (2.3) and (2.1), respectively. Then by manipulating unitarily the oracle selective diagonal operator of (2.3) or the unitary oracle selective diagonal operator of (2.1) one is able to realize an efficient search-space dynamical reduction. Such



a unitary manipulation is generally complex [2, 5a]. It consists of not only the unitary operators (or transformations) but also their inverses. Here the efficient search-space dynamical reduction is no longer described in detail.

An efficient search-space dynamical reduction is the first step to realizing an efficient unstructured quantum search process, i.e., the *HSSS* quantum search process. It eliminates the unstructured search space. Consequently it eliminates the classical effect that is hidden behind the unstructured search space. Therefore, it is the necessary condition for the *HSSS* quantum search process as a whole to achieve a super-square or even an exponential quantum-searching speedup. However, an efficient search-space reduction alone is not sufficient to solve efficiently an unstructured search problem. It can not yet form completely an efficient quantum search process. Instead, its ultimate purpose is to generate the so-called information-carrying unitary operators which further act as the basic building blocks of the quantum-state-difference amplification (QUANS DAM) process at the second step of the *HSSS* quantum search process.

A search-space dynamical reduction is performed in the frame of unitary quantum dynamics. But there may be two equivalent but different pictures [1, 2, 5] to describe it: the unitary state-transformation description and the unitary quantum dynamical description. From the picture of the unitary state transformation an efficient search-space dynamical reduction [2] realizes that the information of one component state (or a few) of the solution state of the search problem is efficiently transferred from the unstructured search space to a polynomially-small subspace (or subset), when the exponentially-large unstructured search space is eliminated. Equivalently, from the picture of the unitary quantum dynamics, this efficient search-space dynamical reduction generates an *information-carrying (IC)* unitary operator from a unitary sequence consisting of the unitary oracle selective diagonal operators and the purely quantum-mechanical (*QM*) unitary operators. This generated *IC* unitary operator carries the information of one component state (or a few) of the solution state. It works in the two polynomially-small subspaces which belong to the math Hilbert space of the search problem and the Hilbert space of the quantum system, respectively. The generated *IC* unitary operator also may be further used to prepare a desired *IC* unitary operator.

The simplest *IC* unitary operator which may be efficiently prepared by the efficient search-space dynamical reduction [2] may be briefly written as

$$U_{\lambda}^{ic}(\theta_m, a_m^s) = \exp(-ia_m^s \theta_m I_{m\lambda}), \quad 1 \leq m \leq n, \quad \lambda = x, y, z \quad (2.9)$$

where the rotating angle  $\theta_m$  is proportional to  $1/2^n$  and hence it is exponentially small. This *IC* unitary operator is called the basic *IC* unitary operator. It is built out of the unitary oracle selective diagonal operators of (2.1) and the *QM* unitary operators. It was first proposed in Ref. [5a]. It carries only the information ( $a_m^s$ ) of the component state  $|s_m\rangle$  of the solution state. The operator  $I_{m\lambda}$  in (2.9) may be a spin operator of the single  $m$ -th spin-1/2 of the  $n$ -qubit spin system. Then in this case the *IC* unitary operator was prepared in Ref. [5], but the preparation could not be efficient. Below consider only the case that the spin operator  $I_{m\lambda}$  of the  $m$ -th

spin-1/2 stands for the pseudospin-1/2 operator of the  $n$ -qubit spin system, which may be written as  $|0_1\rangle\langle 0_1| \otimes \dots \otimes |0_{m-1}\rangle\langle 0_{m-1}| \otimes I_{m\lambda} \otimes |0_{m+1}\rangle\langle 0_{m+1}| \otimes \dots \otimes |0_n\rangle\langle 0_n|$ . Then in this case the  $IC$  unitary operator may be efficiently prepared in Ref. [2]. It works in a two-dimensional subspace ( $a_m^s = \pm 1$ ) of the math Hilbert space of the search problem and also in a two-dimensional subspace  $\{|0_1\rangle \dots |0_{m-1}\rangle |s_m\rangle |0_{m+1}\rangle \dots |0_n\rangle\}$  ( $s_m = 0, 1$ ) of the Hilbert space of the  $n$ -qubit spin system. Note that when the  $IC$  unitary operator is used, the  $n$ -qubit spin system should be set to the initial state  $|0\rangle$  in advance except the  $m$ -th spin-1/2.

As shown in (2.9), the  $IC$  unitary operator depends upon the  $m$ -th spin operator  $I_{m\lambda}$  ( $\lambda = x, y, z$ ). It can affect not only the  $m$ -th spin-1/2 but also any coupled spin system which consists of the  $m$ -th spin-1/2 and other spins that do not belong to the  $n$ -qubit spin system. Moreover, by a suitable unitary transformation the  $m$ -th spin operator  $I_{m\lambda}$  may be changed to another spin operator  $I_{m'\lambda}$  that does not belong to the  $n$ -qubit spin system. Then the generated unitary operator  $U_\lambda^{ic}(\theta_{m'}, a_m^s) = \exp(-ia_m^s \theta_{m'} I_{m'\lambda})$  ( $m' \notin [1, n]$ ) also may act as an  $IC$  unitary operator. It may be used in a spin system that contains the  $m'$ -th spin-1/2 but is different from the  $n$ -qubit spin system, here the  $n$ -qubit spin system should be simply set to the initial state  $|0\rangle$  in advance when the  $IC$  unitary operator is used.

The subspace of the math Hilbert space of the search problem for the  $IC$  unitary operator  $U_\lambda^{ic}(\theta_m, a_m^s)$  is two-dimensional after the search-space dynamical reduction, while the original unstructured search space of the search problem is exponentially large ( $2^n$ -dimensional) before the search-space dynamical reduction. This fact that the  $IC$  unitary operator  $U_\lambda^{ic}(\theta_m, a_m^s)$  does not work in the whole unstructured search space but in its two-dimensional subspace shows clearly that the search-space dynamical reduction indeed eliminates the exponentially large unstructured search space. Now the unstructured property of the search space does not have an effect on the behavior of the  $IC$  unitary operator  $U_\lambda^{ic}(\theta_m, a_m^s)$ . In contrast, it has an essential effect on the behavior of the oracle operation of a conventional quantum search algorithm. This difference also shows that the  $HSSS$  quantum search process is essentially different from a conventional quantum search algorithm.

In the  $HSSS$  quantum search process the basic  $IC$  unitary operator  $U_\lambda^{ic}(\theta_m, a_m^s)$  may act as the basic building block of a UNIDYSLOCK process or a QUANSDAM process. Here an exponential QUANSDAM process is just the second step of the  $HSSS$  quantum search process. Its purpose is to extract the information ( $a_m^s$ ) of the component state  $|s_m\rangle$  of the solution state in an efficient way. A UNIDYSLOCK (or QUANSDAM) process may be constructed in a quantum system such as a single-atom system. Its own  $IC$  unitary operator may be different from system to system. These different  $IC$  unitary operators may be prepared by starting from the basic  $IC$  unitary operator of (2.9).

In next sections one will see that the basic  $IC$  unitary operator is the necessary component for a QUANSDAM (or UNIDYSLOCK) process. Without it, in a QUANSDAM process the quantum-state difference between a pair of non-orthogonal quantum states is not able to build up and the  $QSD$  amplification

ability can not be promoted greatly by the  $QM$  unitary dynamical processes. One may say intuitively that the basic  $IC$  unitary operator provides the seed to grow the quantum-state difference in a QUANSDAM process.

In the quantum-computing speedup theory the principle is general that an essential quantum-computing speedup is achieved by the interaction between the quantum physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) and the mathematical-logical principle that a computational problem obeys. As described above, an efficient search-space dynamical reduction is realized by the interaction between the Hilbert space of a quantum system and the math Hilbert space of an unstructured search problem in the frame of unitary quantum dynamics, while the symmetric structure of the Hilbert space of the quantum system shapes the interaction. Therefore, the principle indeed governs the efficient search-space dynamical reduction. It is not only applied to the first step of the  $HSSS$  quantum search process, i.e., an efficient search-space dynamical reduction, but also it is available over the whole  $HSSS$  quantum search process. There is no exception for a UNIDYSLOCK process and a QUANSDAM process.

### 3. The principle of a UNIDYSLOCK process and a QUANSDAM process

Generally a unitary dynamical state-locking process (or briefly a UNIDYSLOCK process) is defined as such a unitary process that transforms simultaneously two or more orthogonal quantum states to their corresponding non-orthogonal quantum states whose differences may be arbitrarily small [1, 7, 6]. At first the author thought that a UNIDYSLOCK process could be realized by a purely quantum-mechanical unitary dynamical process [7]. Unfortunately that is not correct! After an intense research into it the author realized eventually that a UNIDYSLOCK process does not make sense if one interprets it from the point of view of the purely quantum-mechanical unitary dynamics. Because a UNIDYSLOCK process is not only a quantum-mechanical unitary dynamical process but also it obeys the mathematical-logical principle of the search problem [5, 6, 1, 2], completely and exactly understanding a UNIDYSLOCK process must consider both the aspect of the unitary quantum dynamics and the aspect of the mathematical-logical principle and especially the interaction between the quantum physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) and the mathematical-logical principle. Below a detailed investigation is carried out on the principle and mechanism of how a UNIDYSLOCK process and its inverse process, i.e., a QUANSDAM process work in a quantum system on the basis of the quantum-computing speedup theory [1].

Before a UNIDYSLOCK process is discussed in detail, one needs to set up a quantitative measure for the quantum-state difference between a pair of non-orthogonal quantum states, because the quantum-state-difference ( $QSD$ ) varying rate is the characteristic physical quantity of a UNIDYSLOCK process and a QUANSDAM process. What one discusses in the paper is a pure quantum

system. In quantum mechanics for a pure quantum system the most naive method to measure quantitatively the quantum-state difference between a pair of quantum states is perhaps to use the scalar (or inner) product of the pair of quantum states. For intuition in the paper the scalar product of a pair of wave functions (i.e., quantum states) also is called the overlapping integral of the pair of wave functions. Suppose that  $\varphi_1(x)$  and  $\varphi_2(x)$  are a pair of normalized wave functions in one-dimensional coordinate space  $(-\infty, +\infty)$ . Then in quantum mechanics the scalar product of the pair of wave functions may be defined by [14, 17]

$$\rho_{12} = \langle \varphi_1 | \varphi_2 \rangle = \int_{-\infty}^{\infty} \varphi_1(x)^* \varphi_2(x) dx \quad (3.1)$$

The absolute scalar product  $|\rho_{12}|$  satisfies  $0 \leq |\rho_{12}| \leq 1$ . Its physical meaning is explained as follows. If  $|\rho_{12}| = 1$ , then both the wave functions  $\varphi_1(x)$  and  $\varphi_2(x)$  are the same one up to a phase factor. If  $|\rho_{12}| = 0$ , then both the wave functions are orthogonal to one another. If  $0 < |\rho_{12}| < 1$ , both the wave functions are neither equal to nor orthogonal to one another and instead they are non-orthogonal to one another. Thus, the scalar product (or the overlapping integral)  $\rho_{12}$  may be used to measure quantitatively the quantum-state difference between a pair of wave functions. As a typical example, the wave functions  $\varphi_1(x)$  and  $\varphi_2(x)$  are taken as the two standard Gaussian wave-packet states [15], respectively, each one of which has the characteristic parameters  $\{x_l, p_l, (\Delta x_l)^2, T_l\}$  for  $l = 1, 2$ . Then it can turn out that the absolute overlapping integral  $|\rho_{12}|$  may be written as

$$|\rho_{12}| = \left( \frac{4\Delta_1^2 \Delta_2^2}{(\Delta_1^2 + \Delta_2^2)^2 + \beta_{12}^2} \right)^{1/4} \times \exp \left( \frac{D'}{(\Delta_1^2 + \Delta_2^2)^2 + \beta_{12}^2} \right)$$

with

$$D' = -\frac{1}{\hbar^2} p_{12}^2 \Delta_1^2 \Delta_2^2 (\Delta_1^2 + \Delta_2^2) - \frac{1}{4} \Delta_1^2 \left( x_{12} - \frac{2}{\hbar} p_{12} \beta_2 \right)^2 - \frac{1}{4} \Delta_2^2 \left( x_{12} - \frac{2}{\hbar} p_{12} \beta_1 \right)^2$$

where  $\Delta_l^2 = (\Delta x_l)^2$ ,  $\beta_l = (\frac{\hbar T_l}{2m})$ ;  $p_{12} = p_1 - p_2$ ,  $x_{12} = x_1 - x_2$ , and  $\beta_{12} = \beta_1 - \beta_2$ .

Below it first proves a particularly important result that a purely quantum-mechanical unitary dynamical process cannot change two or more orthogonal quantum states to their corresponding non-orthogonal states at the same time. According to quantum mechanics [14, 17] it can turn out that a pair of quantum states keep their scalar product unchanged under action of an arbitrary unitary transformation. Suppose that a pair of initial quantum states  $|\Psi_k\rangle$  and  $|\Psi_l\rangle$  which may be arbitrary are acted on by an arbitrary unitary transformation at the same time:

$$U : \begin{cases} |\Psi_k\rangle \rightarrow |\varphi_k\rangle \\ |\Psi_l\rangle \rightarrow |\varphi_l\rangle \end{cases} \quad (3.2)$$

Then the scalar product of the final two states  $|\varphi_k\rangle$  and  $|\varphi_l\rangle$  satisfies the relation:

$$\langle \varphi_k | \varphi_l \rangle = (\langle \Psi_k | U^\dagger)(U | \Psi_l \rangle) = \langle \Psi_k | \Psi_l \rangle. \quad (3.3)$$

This shows that the scalar product of the final two states is equal to that one of the initial two states. Therefore, an arbitrary unitary transformation cannot change the scalar product of a pair of quantum states. This is a universal conclusion in quantum mechanics [14, 17]. It directly leads to that a purely quantum-mechanical unitary dynamical process can not transform simultaneously two or more orthogonal states to their corresponding non-orthogonal states.

It is well-known in quantum mechanics that a quantum system obeys the unitary quantum dynamics. (Throughout the paper quantum measurement is merely used as a tool to obtain the final computational result at the end of a quantum computational process. This is required by the quantum-computing speedup theory [1]. Therefore, here quantum measurement is not considered.) Then, as deduced from (3.2) and (3.3), in any unitary time evolution process of a quantum system the quantum-state difference between a pair of quantum states of the quantum system is never changed. This means that the quantum-state difference is never changed during the unitary time evolution process, although any one of the pair of quantum states may be a superposition state, a quantum entanglement state, or a coherent state and so on. This shows clearly that *any quantum-state effects in a quantum system can not cause varying of quantum-state difference during a unitary quantum dynamical process, here the quantum-state effects may include the quantum-state superposition, coherence interference, entanglement and nonlocal effect (i.e., spooky action at a distance), correlation and so on.* This is the first important property of the quantum-state-difference varying.

There is no doubt that in quantum mechanics a UNIDYSLOCK process obeys the unitary quantum dynamics. Then according to (3.2) and (3.3) it can not change the scalar product of a pair of quantum states and hence it is not able to transform simultaneously two or more orthogonal states to their corresponding non-orthogonal states. However, according to its definition a UNIDYSLOCK process is such a unitary quantum dynamical process that can change two or more orthogonal states to their corresponding non-orthogonal states at the same time. This means that a UNIDYSLOCK process does not make sense in pure quantum mechanics.

As a quantum computational process (or sub-process) a UNIDYSLOCK process obeys not only the unitary quantum dynamics but also the mathematical-logical principle of the search problem [6, 7, 1]. It is substantially different from a conventional quantum-mechanical unitary dynamical process in that it must obey the mathematical-logical principle of the search problem. Thus, both the aspects of the unitary quantum dynamics and the mathematical-logical principle as well as the interaction between the quantum physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) and the mathematical-logical principle are key to understanding exactly and completely how a UNIDYSLOCK process works in a quantum system. It must be from the point of view of the mathematical-logical principle to comprehend that a UNIDYSLOCK process is able to change the scalar product of a pair of quantum states. It must be understood that the interaction can cause the quantum-state-difference varying and may influence greatly the  $QSD$  varying

rate of a UNIDYSLOCK (or QUANSDAM) process.

In general, a UNIDYSLOCK (or QUANSDAM) process consists of the information-carrying unitary operators (i.e., the *IC* unitary operators) and the purely quantum-mechanical unitary operators (i.e., the *QM* unitary operators). Its core is the *IC* unitary operators. Its characteristic property that a UNIDYSLOCK process can change the scalar product of a pair of quantum states (i.e., the quantum-state difference between the two states) is uniquely original from the *IC* unitary operators. As shown in (3.2) and (3.3) above, this property has nothing to do with any *QM* unitary operator. A UNIDYSLOCK process could be best understood by performing its inverse process, i.e., a QUANSDAM process in a quantum system. For simplicity, here consider that the QUANSDAM process consists of only one basic *IC* unitary operator of (2.9) and it is applied to a single spin-1/2 (or pseudospin-1/2) system. The spin system has only two orthonormal basis states  $|0\rangle$  and  $|1\rangle$ . Suppose that at the initial time the spin system is in the state  $|0\rangle$ . Then it is acted on by the basic *IC* unitary operator of (2.9). One therefore obtains the simple QUANSDAM process in the system:

$$U_x^{ic}(\theta_m, a_m^s)|0\rangle = \cos \frac{1}{2}\theta_m|0\rangle - ia_m^s \sin \frac{1}{2}\theta_m|1\rangle. \quad (3.4)$$

It seems that from the point of view of pure quantum mechanics there is not a big difference between the QUANSDAM process and a usual unitary state transformation with the (time or phase) parameter  $(a_m^s \theta_m)$ . This is a trivial understanding for a QUANSDAM process. It is too trivial to capture the essential aspect of a QUANSDAM process. However, this simple QUANSDAM process may be comprehended more exactly and more deeply from the point of view of the mathematical-logical principle. As pointed out in the preceding section, the basic *IC* unitary operator  $U_x^{ic}(\theta_m, a_m^s)$  works in the two-dimensional subspace  $(a_m^s = \pm 1)$  of the math Hilbert space of the search problem. Then at the end of the QUANSDAM process there must be two possible states to be generated in the spin system, although only one of the two states really appears in the system. These two possible states may be written as

$$U_x^{ic}(\theta_m, a_m^s)|0\rangle = \begin{cases} |\psi_{+1}\rangle, & \text{if } a_m^s = +1 \\ |\psi_{-1}\rangle, & \text{if } a_m^s = -1 \end{cases} \quad (3.5)$$

where  $|\psi_{\pm 1}\rangle = \cos \frac{1}{2}\theta_m|0\rangle \mp i \sin \frac{1}{2}\theta_m|1\rangle$ . It may be thought that both the final states  $|\psi_{+1}\rangle$  and  $|\psi_{-1}\rangle$  are generated from the initial two states which are the same and equal to the state  $|0\rangle$  by the QUANSDAM process, respectively. Now one may examine the quantum-state difference between the final two states  $|\psi_{+1}\rangle$  and  $|\psi_{-1}\rangle$ . It turns out that the scalar product of the final two states is given by

$$\langle \psi_{+1} | \psi_{-1} \rangle = \cos \theta_m. \quad (3.6)$$

When  $\theta_m = 0$ , both the states  $|\psi_{+1}\rangle$  and  $|\psi_{-1}\rangle$  are the same one and equal to the initial state  $|0\rangle$ . Then the scalar product  $\langle \psi_{+1} | \psi_{-1} \rangle$  is equal to one. However, as shown in (3.6), when  $\theta_m \neq 0$ , the scalar product may not equal one. Particularly, when  $\theta_m = \pi/2$ , it is equal to zero and hence both the states  $|\psi_{+1}\rangle$

and  $|\psi_{-1}\rangle$  are orthogonal to each other. These show clearly that the QUANS-DAM process is able to change the scalar product of the two states  $|\psi_{+1}\rangle$  and  $|\psi_{-1}\rangle$  and so is the UNIDYSLOCK process, i.e., the inverse of the QUANS-DAM process. It is clear that under a UNIDYSLOCK process the absolute scalar product  $|\langle\psi_{+1}|\psi_{-1}\rangle|$  is changed from zero to one. Conversely, under a QUANS-DAM process it is changed from one to zero. The above investigation for the simple QUANS-DAM process highlights the most basic characteristic feature of a general QUANS-DAM (or UNIDYSLOCK) process that a QUANS-DAM (or UNIDYSLOCK) process is able to change the quantum-state difference between a pair of quantum states.

Here it needs to emphasize the unitarity of a UNIDYSLOCK process. Without unitarity, a UNIDYSLOCK process will be trivial in quantum computation (as explained below). It therefore rules out any irreversible physical processes including quantum measurement and any irreversible mathematical-logical operations as its building blocks. It is well known that two or more orthogonal quantum states may be changed to a quantum state simultaneously by a non-equilibrium irreversible process (i.e., a non-unitary dynamical process). This is the only non-trivial physical process that can change the quantum-state difference of a pair of quantum states except the UNIDYSLOCK (or QUANS-DAM) process. If now a UNIDYSLOCK process contained a non-equilibrium irreversible sub-process, then certainly it would be able to change many orthogonal states to their corresponding non-orthogonal states at the same time. In this case the UNIDYSLOCK process is not significantly different from a usual non-equilibrium irreversible process. Hence it becomes trivial in quantum computation. Here the non-equilibrium irreversible sub-process may be purely physical or it contains any irreversible mathematical-logical operation. In either case it veils the essential aspect of a UNIDYSLOCK process. It results in that one cannot understand exactly the essential aspect of a UNIDYSLOCK process. A non-equilibrium irreversible process could make a UNIDYSLOCK process more quickly changing many orthogonal states to their corresponding non-orthogonal states at the same time, but it also could make the inverse of the UNIDYSLOCK process more hard amplifying the quantum-state difference. Thus, it could degrade or even destroy the *QSD* amplification ability of a QUANS-DAM process. Beside this problem the energy dissipation problem of irreversibility (See, e.g., Ref. [20]) still needs to be considered for a quantum computation to use a non-equilibrium irreversible process as its building block.

A QUANS-DAM (or UNIDYSLOCK) process generally contains many basic *IC* unitary operators of (2.9) and *QM* unitary operators. As shown in (3.2) and (3.3), any *QM* unitary operator of the process can not change the scalar product of a pair of quantum states. Then only the basic *IC* unitary operators of the process are able to change the scalar product of a pair of quantum states (See, e.g.,  $\langle\psi_{+1}|\psi_{-1}\rangle$  in (3.6)). That is, only the basic *IC* unitary operators can cause the quantum-state-difference varying. As pointed out in the preceding section, the basic *IC* unitary operator is built out of the unitary oracle selective diagonal operators of (2.1) and the *QM* unitary operators. Again here the *QM* unitary operators can not change the quantum-state difference. Therefore, the

quantum-state-difference varying is uniquely original from the unitary oracle selective diagonal operator.

As shown in the section 5, the unitary oracle selective diagonal operator works in both the Hilbert space of the quantum system and the math Hilbert space of the search problem. In the Hilbert space it is a single reversible black-box functional operation, while in the math Hilbert space it may be considered as a mathematical-parallel reversible black-box functional operation as a whole. These reversible black-box functional operations (or the component functional operations) in both the Hilbert spaces altogether characterize completely the mathematical-logical principle of the unstructured search problem, indicating that the unitary oracle selective diagonal operator can characterize faithfully the mathematical-logical principle of the search problem. One may say that the unitary oracle selective diagonal operator is a reversible black-box functional operation of the search problem. *This reversible black-box functional operation of the unstructured search problem really causes the change of the quantum-state difference in a UNIDYSLOCK (or QUANSDAM) process.*<sup>7</sup> This is perhaps a most mysterious result for the Lecerf-Bennett reversible computational theory [8]. This result is obtained on the basis of the quantum-computing speedup theory [1].

As pointed out in the section 2, the basic *IC* unitary operator of (2.9) is generated by the search-space dynamical reduction. The latter is realized by the interaction between the Hilbert space of the quantum system and the math Hilbert space of the search problem in the frame of unitary quantum dynamics. Thus, the unitary quantum dynamics, the Hilbert space of the quantum system with its own symmetric structure, and the mathematical-logical principle of the search problem may affect in principle the basic *IC* unitary operator. Consider that the basic *IC* unitary operator works in a two-dimensional reduction subspace of the math Hilbert space and a two-dimensional subspace of the Hilbert space of the quantum system and both the subspaces each have a minimum dimensional size. Then the effect of the Hilbert-space symmetric structure could be secondary on the basic *IC* unitary operator. Therefore, the basic *IC* unitary operator is affected mainly by the unitary quantum dynamics, the Hilbert space of the quantum system, and the mathematical-logical principle. Consider further that the pure unitary quantum dynamics alone can not change the quantum-state difference. Then these show that *the quantum-state-difference varying driven by the basic IC unitary operator is caused by the interaction between the unitary quantum dynamics and the mathematical-logical principle and also the interaction between the Hilbert space of the quantum system and the math Hilbert space of the search problem.* This is the second important property of the quantum-state-difference varying. This property also could be inferred directly from the basic *IC* unitary operator itself.

The property is very important that the effect of the symmetric structure of the Hilbert space of the quantum system could be secondary on the quantum-

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<sup>7</sup>The reversible black-box functional operation also is called the computational resource of a UNIDYSLOCK (or QUANSDAM) process [2].



state-difference varying that is driven by the basic  $IC$  unitary operators of (2.9). It implies that a single quantum system such as a single-atom system that does not have the symmetric structure of the Hilbert space of a composite quantum system could be appropriate to realize an exponential QUANSDAM process whose basic building block is the basic  $IC$  unitary operator.

Below it is shown more clearly that the quantum-state-difference varying driven by the basic  $IC$  unitary operator is still caused by the interaction between the Hilbert space of the quantum system and the math Hilbert space of the search problem, although the effect of the symmetric structure of the Hilbert space of the quantum system could be secondary on the quantum-state-difference varying.

As shown in (3.2) and (3.3), in quantum mechanics it is impossible that many orthogonal states of a quantum system are simultaneously changed to their corresponding non-orthogonal states by a unitary quantum dynamical process. A UNIDYSLOCK process is a unitary quantum dynamical process. A mysterious thing is that by the UNIDYSLOCK process many orthogonal quantum states may be transformed simultaneously to their corresponding non-orthogonal states. Here it should be pointed out that those orthogonal states that are simultaneously changed to their corresponding non-orthogonal states by a UNIDYSLOCK process are not all in the same Hilbert space of the quantum system but in the joint Hilbert space which consists of the Hilbert space of the quantum system and the math Hilbert space of the search problem. This can be best illustrated with the simple QUANSDAM process of (3.5).

It is known from (3.5) that there are two possible quantum states (i.e., the states  $|\psi_{+1}\rangle$  and  $|\psi_{-1}\rangle$ ) at the end of the QUANSDAM process of (3.4). One quantum state (either the state  $|\psi_{+1}\rangle$  or  $|\psi_{-1}\rangle$ ) is in the two-state Hilbert space  $\{|0\rangle, |1\rangle\}$  of the single spin-1/2 quantum system (i.e., the physical state space), while another is not in the Hilbert space  $\{|0\rangle, |1\rangle\}$  but in the two-dimensional subspace of the math Hilbert space of the search problem (i.e., the unphysical state space). Both the states  $|\psi_{+1}\rangle$  and  $|\psi_{-1}\rangle$  can not belong to the same Hilbert space  $\{|0\rangle, |1\rangle\}$  at the same time. If a QUANSDAM process consists of many steps, each step containing one basic  $IC$  unitary operator, then it is impossible that at the end of one step the state  $|\psi_{+1}\rangle$  appears in the Hilbert space  $\{|0\rangle, |1\rangle\}$ , while at the end of another step the state  $|\psi_{-1}\rangle$  appears in the same subspace. The only exception is that both the states  $|\psi_{+1}\rangle$  and  $|\psi_{-1}\rangle$  are the same one. The reason for these is that the solution state of the search problem is unique and deterministic and the information conservation law must be obeyed (See also the next paragraph). Both the states  $|\psi_{+1}\rangle$  and  $|\psi_{-1}\rangle$  should be considered as the members of the two-dimensional subspace of the math Hilbert space, because one does not know in advance which one of the two states does not appear in the Hilbert space  $\{|0\rangle, |1\rangle\}$  in the mathematical-logical meaning of search. The physical behavior of the quantum state that appears in the Hilbert space  $\{|0\rangle, |1\rangle\}$  of the quantum system is characterized completely by the quantum physical laws. This quantum state is the joint point between both the Hilbert space of the quantum system and the math Hilbert space of the search problem. It is really responsible for extracting the information ( $a_m^s$ ) of the  $m$ -th component

state of the solution state. On the other hand, the existence of the math Hilbert space is necessary, so that the QUANSDAM process of (3.4) is able to change the quantum-state difference between the pair of quantum states  $|\psi_{+1}\rangle$  and  $|\psi_{-1}\rangle$ . Thus, the math Hilbert space is necessary to realize a general QUANSDAM (or UNIDYSLOCK) process.

Apparently the simple QUANSDAM process of (3.5) performs the conditional unitary operation. That is, if  $a_m^s = +1$ , it performs the unitary operation  $U_x^{ic}(\theta_m, a_m^s)|0\rangle = |\psi_{+1}\rangle$  and if  $a_m^s = -1$ , it performs the unitary operation  $U_x^{ic}(\theta_m, a_m^s)|0\rangle = |\psi_{-1}\rangle$ . Actually it may be considered that this simple QUANSDAM process performs a mathematical-parallel black-box functional operation as a whole. Note that the basic *IC* unitary operator  $U_x^{ic}(\theta_m, a_m^s)$  of (2.9), which may be considered as a black-box functional operation, works in both the two-dimensional subspace ( $a_m^s = \pm 1$ ) of the math Hilbert space and the Hilbert space of the single spin-1/2 quantum system. Then this mathematical-parallel black-box functional operation owns two component functional operations. One component functional operation that carries the information of the component state of the real solution state is performed unambiguously in the Hilbert space of the quantum system, while another that carries the information of the component state of the candidate solution states is performed unambiguously in the subspace of the math Hilbert space but not in the Hilbert space of the quantum system. It may be thought that these two component functional operations belong to the same subspace of the math Hilbert space but one of which belongs to the quantum system and another does not. It therefore may be considered that the QUANSDAM process of (3.5) performs a mathematical-parallel black-box functional operation in the subspace of the math Hilbert space. A mathematical-parallel functional operation is essentially different from a conventional quantum parallel operation [11, 10, 9]. It does not obey the (linear) superposition principle in quantum physics. Each component functional operation of a mathematical-parallel functional operation obeys the information conservation law independently. The present analysis is available not only for the simple QUANSDAM process of (3.5) but also for a general QUANSDAM process.

The above analysis shows that a QUANSDAM (or UNIDYSLOCK) process could change the quantum-state difference between a pair of quantum states only when the math Hilbert space of the search problem is involved. Therefore, the change of the quantum-state difference between any two states, which are in the Hilbert space of the quantum system and the math Hilbert space of the search problem, respectively, during a QUANSDAM (or UNIDYSLOCK) process is involved in the interaction between the Hilbert space of the quantum system and the math Hilbert space of the search problem. This is part of the second property of the quantum-state-difference varying (See above).

By combining the first and second properties of the quantum-state-difference varying one can conclude that *the quantum-state-difference varying in a QUANSDAM (or UNIDYSLOCK) process is caused only by the interaction between the quantum physical laws (i.e., the unitary quantum dynamics and the Hilbert space of the quantum system) and the mathematical-logical principle of the search prob-*

lem.

The above investigation into a UNIDYSLOCK (or QUANSDAM) process is limited to the simple case that the reduction subspace of the math Hilbert space is two-dimensional. Actually the dimension of the reduction subspace may be more than two in a general QUANSDAM (or UNIDYSLOCK) process. Then in the general case there are more than two candidate quantum states at the end of the QUANSDAM (or UNIDYSLOCK) process. Generally, there are  $\mathbb{N}$  candidate quantum states at the end of a QUANSDAM (or UNIDYSLOCK) process if the reduction subspace is  $\mathbb{N}$ -dimensional. However, among these  $\mathbb{N}$  states there is only one quantum state in the Hilbert space of the quantum system and this quantum state carries the information of the solution state, because there is only one solution state to the search problem. Then the remaining  $\mathbb{N} - 1$  candidate states all are in the reduction subspace of the math Hilbert space. Just like the simple QUANSDAM process of (3.5) this general QUANSDAM process contains a mathematical-parallel black-box functional operation of the reduction subspace of the math Hilbert space.

A typical example is given below. Suppose that an  $n$ -qutrit quantum system is used to solve the unstructured search problem and that after the search-space dynamical reduction the basic *IC* unitary operator in form is still given by  $U_x^{ic}(\theta_m, a_m^s)$  of (2.9) with the triple-value logical number  $a_m^s = +1, 0, -1$  (See the section 2). This really means that the original unstructured search space of the search problem is dynamically reduced to a three-dimensional reduction subspace of the math Hilbert space of the search problem. Then in the present case the simple QUANSDAM process of (3.5) should be modified to the form

$$U_x^{ic}(\theta_m, a_m^s)|0\rangle = \begin{cases} |\psi_{+1}\rangle, & \text{if } a_m^s = +1 \\ |\psi_0\rangle, & \text{if } a_m^s = 0 \\ |\psi_{-1}\rangle, & \text{if } a_m^s = -1 \end{cases} \quad (3.7)$$

where the initial computational basis state  $|0\rangle$  is equal to the state  $|+1\rangle$  (See the section 2). This shows that there are three candidate quantum states at the end of the QUANSDAM process. Here only one quantum state is in the Hilbert space of the  $n$ -qutrit quantum system, while the rest two candidate states are in the three-dimensional reduction subspace. This is different from the case in the  $n$ -qubit quantum system above. It may be considered that the QUANSDAM process of (3.7) performs a mathematical-parallel black-box functional operation in the three-dimensional reduction subspace of the math Hilbert space.

It seems that the larger the reduction subspace of the math Hilbert space, the weaker the interaction between the Hilbert space of the quantum system and the math Hilbert space of the search problem and the smaller the *QSD* amplification ability for a QUANSDAM process. This should be true for a large reduction subspace of the math Hilbert space.

A QUANSDAM process like (3.4) can change the scalar product of a pair of states, one of which is in the Hilbert space and another in the math Hilbert space. However, it can prove below that a QUANSDAM (or UNIDYSLOCK)

process can not change the scalar product of any two quantum states in the Hilbert space of a quantum system. In general a quantum state  $|\psi_k\rangle$  of the quantum system could become a function of the unit number  $b_m^s = \pm 1$  during a QUANSDAM (or UNIDYSLOCK) process. Then generally one has  $|\psi_k\rangle = |\psi_k(b_m^s)\rangle$ . Note that in the mathematical-logical meaning of search one does not know in advance which one of the states  $|\psi_k(+1)\rangle$  and  $|\psi_k(-1)\rangle$  belongs to the quantum system. Thus, one must consider both the cases  $b_m^s = +1$  and  $b_m^s = -1$ , respectively. Suppose that there are any two states  $|\psi_k(b_m^s)\rangle$  and  $|\psi_l(b_m^s)\rangle$  of the quantum system with the unit number  $b_m^s$  equal to some logical value (either  $+1$  or  $-1$ ). When both the states are acted on by the basic *IC* unitary operator  $U_\lambda^{ic}(\theta_m, a_m^s)$  of (2.9) at the same time, one obtains  $U_\lambda^{ic}(\theta_m, a_m^s)|\psi_k(b_m^s)\rangle = |\varphi_k(a_m^s, b_m^s)\rangle$  and  $U_\lambda^{ic}(\theta_m, a_m^s)|\psi_l(b_m^s)\rangle = |\varphi_l(a_m^s, b_m^s)\rangle$ . Now the scalar product of the two states  $|\varphi_k(a_m^s, b_m^s)\rangle$  and  $|\varphi_l(a_m^s, b_m^s)\rangle$  is given by

$$\begin{aligned} & \langle \varphi_k(a_m^s, b_m^s) | \varphi_l(a_m^s, b_m^s) \rangle \\ &= (\langle \psi_k(b_m^s) | U_\lambda^{ic}(\theta_m, a_m^s)^+ ) U_\lambda^{ic}(\theta_m, a_m^s) | \psi_l(b_m^s) \rangle \\ &= \langle \psi_k(b_m^s) | \psi_l(b_m^s) \rangle \end{aligned}$$

where the fixed logical value  $a_m^s = +1$  or  $-1$ . This formula shows that the scalar product of the two states  $|\varphi_k(a_m^s, b_m^s)\rangle$  and  $|\varphi_l(a_m^s, b_m^s)\rangle$  is equal to the one of the two states  $|\psi_k(b_m^s)\rangle$  and  $|\psi_l(b_m^s)\rangle$ , indicating that the quantum-state difference between the two states  $|\psi_k(b_m^s)\rangle$  and  $|\psi_l(b_m^s)\rangle$  of the quantum system can not be changed by the basic *IC* unitary operator  $U_\lambda^{ic}(\theta_m, a_m^s)$ . It can further prove that the quantum-state difference between any two states of the quantum system can not be changed by a general QUANSDAM (or UNIDYSLOCK) process that consists of the basic *IC* unitary operators of (2.9) and the *QM* unitary operators. This result leads to an extension to the first property of the quantum-state-difference varying. As shown above, for any fixed logical value  $a_m^s = +1$  or  $-1$  the basic *IC* unitary operator  $U_\lambda^{ic}(\theta_m, a_m^s)$  may be really considered as a *QM* unitary operator, when it is applied to a quantum system. Then one can deduce from (3.2) and (3.3) that the quantum-state difference between any two states of the quantum system never changes under action of the basic *IC* unitary operator on the quantum system or more generally under action of a general QUANSDAM (or UNIDYSLOCK) process. Therefore, during the QUANSDAM (or UNIDYSLOCK) process any quantum-state effect of the quantum system can not cause the change of the quantum-state difference.

That a UNIDYSLOCK (or QUANSDAM) process is able to change the quantum-state difference between a pair of quantum states does not necessarily mean that the UNIDYSLOCK (or QUANSDAM) process can change quickly (e.g., exponentially fast) the quantum-state difference. How fast a UNIDYSLOCK (or QUANSDAM) process can change the quantum-state difference may be measured by the quantum-state-difference (*QSD*) varying rate of the process. For a QUANSDAM process it may be measured conveniently by the *QSD* amplification ability of the process. A general QUANSDAM process consists of the basic *IC* unitary operators and the *QM* unitary operators in addition to

the initial state (often the state is omitted without confusion). Here only the basic *IC* unitary operators can change the quantum-state difference, while the *QM* unitary operators can not. However, without the *QM* unitary operators the basic *IC* unitary operators alone can not lead to a large *QSD* amplification ability and as shown below, they generate only a linear *QSD* amplification ability. Then a super-linear or even an exponential *QSD* amplification ability must be involved in the *QM* unitary operators in a QUANSDAM process. An exponential *QSD* amplification ability for a QUANSDAM process is really crucial to realize an exponential quantum-searching speedup for the *HSSS* quantum search process.

A general QUANSDAM process (omitting the initial state) may be written as

$$QUANSDAM(K, a_m^s) = U_K V_K^{ic}(a_m^s) U_{K-1} \dots U_1 V_1^{ic}(a_m^s) U_0 \quad (3.8)$$

where  $\{U_k\}$  are the *QM* unitary operators and  $\{V_k^{ic}(a_m^s)\}$  the *IC* unitary operators. Without losing generality, the initial state of (3.8) may be simply set to the ground state of the quantum system. The *IC* unitary operator  $V_k^{ic}(a_m^s)$  in (3.8) may take the basic *IC* unitary operator of (2.9), the basic one in (3.7), or more generally an *IC* unitary propagator (See the section 4 below) which consists of the basic *IC* unitary operators and the *QM* unitary operators. Although the *QM* unitary operators  $\{U_k\}$  in (3.8) act on the quantum system only, it may be considered that they are applied to any state of the Hilbert space of the quantum system or any state of the math Hilbert space in a mathematical-parallel form, which is dependent on the logical number  $a_m^s$  in the applied quantum state. The *QSD* amplification ability of the QUANSDAM process of (3.8) could be dependent on several factors. The first factor is the dimensional size of the reduction subspace of the unstructured search space in which the basic *IC* unitary operator works. The second is the quantum system used to realize the QUANSDAM process. More importantly the third is the mutual cooperation between the *IC* unitary operators  $\{V_k^{ic}(a_m^s)\}$  and the *QM* unitary operators  $\{U_k\}$ . Any *QM* unitary operator like  $U_k$  can not change the quantum-state difference and the basic *IC* unitary operator can change only an exponentially-small value of quantum-state difference (See (3.6) above), but the QUANSDAM process of (3.8) could greatly change the quantum-state difference. This implies that the mutual cooperation between the *IC* unitary operators  $\{V_k^{ic}(a_m^s)\}$  and the *QM* unitary operators  $\{U_k\}$  may be very important for the QUANSDAM process to achieve a super-square or even an exponential *QSD* amplification ability. It seems that when the reduction subspace is large, the mutual cooperation becomes uneasy to realize. Therefore, the larger the reduction subspace, the weaker the *QSD* amplification ability of the QUANSDAM process. When the reduction subspace is exponentially large, it should be impossible to achieve an exponential *QSD* amplification ability for the QUANSDAM process no matter what these *QM* unitary operators  $\{U_k\}$  are taken. These properties have nothing to do with any initial state of the QUANSDAM process. Therefore, one must use a small reduction subspace to realize the QUANSDAM process.

The purpose to investigate the QUANSDAM process of (3.8) for the given

$IC$  unitary operators  $\{V_k^{ic}(a_m^s)\}$  is to find an appropriate quantum system and the appropriate  $QM$  unitary operators  $\{U_k\}$  of the quantum system to realize the QUANSDAM process so that the QUANSDAM process can achieve a super-square or even an exponential  $QSD$  amplification ability.

When  $V_k^{ic}(a_m^s)$  in (3.8) is taken as the basic  $IC$  unitary operator  $U_\lambda^{ic}(\theta_m, a_m^s)$  of (2.9), which works in a two-dimensional reduction subspace, there are only two candidate final states in the QUANSDAM process of (3.8) with the initial normalized state  $|\Psi_0\rangle$ :

$$U_K U_\lambda^{ic}(\theta_m, a_m^s) U_{K-1} \dots U_1 U_\lambda^{ic}(\theta_m, a_m^s) U_0 |\Psi_0\rangle = \begin{cases} |\Psi_{+1}^K\rangle, & \text{if } a_m^s = +1 \\ |\Psi_{-1}^K\rangle, & \text{if } a_m^s = -1 \end{cases} \quad (3.9)$$

Here one may simply set the initial state  $|\Psi_0\rangle$  to the ground state of the quantum system. During the QUANSDAM process two intermediate (different) states  $|\Psi_{+1}^k\rangle = QUANSDAM(k, +1) |\Psi_0\rangle$  and  $|\Psi_{-1}^k\rangle = QUANSDAM(k, -1) |\Psi_0\rangle$  with  $1 \leq k \leq K$  forever exclude one another in the quantum system no matter what these  $QM$  unitary operators  $\{U_k\}$  take, where  $QUANSDAM(l, a_m^s)$  is given by (3.8) with the settings  $V_k^{ic}(a_m^s) = U_\lambda^{ic}(\theta_m, a_m^s)$  and  $K = l$ . That is, only one of the two states is in the Hilbert space of the quantum system, while another is in the math Hilbert space of the search problem but is never in the quantum system. Therefore, in the sense of quantum physics there does not exist any real quantum-state effect such as the quantum-state superposition, coherence interference, entanglement and nonlocal effect, or correlation between the two states  $|\Psi_{+1}^k\rangle$  and  $|\Psi_{-1}^k\rangle$  over the whole QUANSDAM process. Here it must be emphasized that this result is universal for a general QUANSDAM (or UNIDYSLOCK) process. Suppose that the reduction subspace in which a general QUANSDAM (or UNIDYSLOCK) process works is  $\mathbb{N}$ -dimensional. Then there should be  $\mathbb{N}$  candidate final states rather than two candidate final states in (3.9). Moreover, only one of these  $\mathbb{N}$  candidate states appears in the quantum system, while the remaining  $\mathbb{N} - 1$  candidate states do not. Clearly there does not exist any real quantum-state effect between the candidate state that appears in the quantum system and any one of the remaining  $\mathbb{N} - 1$  candidate states. Because these remaining  $\mathbb{N} - 1$  candidate states are not in the quantum system, there is not any real quantum-state effect between any two of the remaining  $\mathbb{N} - 1$  candidate states. Therefore, one arrives at the result that there is not any real quantum-state effect between any two of these  $\mathbb{N}$  candidate states in the general QUANSDAM (or UNIDYSLOCK) process. This result is important. By combining this result with the first property of quantum-state-difference varying (See above the property and its extension) one can conclude that *no quantum-state effect of a quantum system can cause the change of the quantum-state difference between the candidate state that appears in the quantum system and any other candidate state of the math Hilbert space during a QUANSDAM (or UNIDYSLOCK) process.*

How to extract the information  $(a_m^s)$  of the component state of the solution state at the end of the QUANSDAM process of (3.9) by quantum measurement? In general, the solution information  $(a_m^s)$  can be obtained only by

distinguishing one of the two candidate states  $|\Psi_{+1}^K\rangle$  and  $|\Psi_{-1}^K\rangle$  from another at the end of the QUANS DAM process. But here it must be pointed out that a single quantum state (either  $|\Psi_{+1}^K\rangle$  or  $|\Psi_{-1}^K\rangle$ ) of the quantum system is not sufficient to obtain the solution information ( $a_m^s$ ). Therefore, when both the states  $|\Psi_{+1}^K\rangle$  and  $|\Psi_{-1}^K\rangle$  are orthogonal to one another, the solution information ( $a_m^s$ ) can be obtained unambiguously by the quantum measurement at the end of the QUANS DAM process. Here these two orthogonal states could be first transferred to the eigenstates of the observable Hermitian operator (i.e., a dynamical variable), respectively. Then by quantum measuring eigenvalue of the observable operator one is able to distinguish these two orthogonal states from one another. In this case the quantum measurement is precise and may be considered to be deterministic. In quantum mechanics it obeys the quantum-measurement postulate that one (or another) of the eigenvalues of a dynamical variable is the only possible value of a precise measurement of the dynamical variable [14, 17]. There is the possibility that both the states  $|\Psi_{+1}^K\rangle$  and  $|\Psi_{-1}^K\rangle$  are not orthogonal to one another. Then in this case in technique it is still possible to use the quantum measurement such as the POVM measurement [28] to distinguish these two non-orthogonal states from one another. Such a quantum measurement is still probabilistic and obeys the Born's rule [17, 14].

It may be thought that the unitary time evolution process of the QUANS DAM process of (3.9) owns two different branches, which correspond to the final two candidate states  $|\Psi_{+1}^K\rangle$  and  $|\Psi_{-1}^K\rangle$ , respectively. These two branches form a mathematical-parallel quantum computation. They are given by ( $1 \leq k \leq K$ )

$$|\Psi_{+1}^k\rangle = U_k U_\lambda^{ic}(\theta_m, +1) U_{k-1} \dots U_1 U_\lambda^{ic}(\theta_m, +1) U_0 |\Psi_0\rangle, \quad (3.10a)$$

$$|\Psi_{-1}^k\rangle = U_k U_\lambda^{ic}(\theta_m, -1) U_{k-1} \dots U_1 U_\lambda^{ic}(\theta_m, -1) U_0 |\Psi_0\rangle. \quad (3.10b)$$

Here only one of the two branches really describes the time evolution process of the quantum system, while another describes the time evolution process of the candidate state that belongs to the reduction subspace of the math Hilbert space. Both are unitary. But the former is a real unitary quantum dynamical process, while the latter is a mathematical unitary dynamical process. Both the candidate states  $|\Psi_{+1}^k\rangle$  and  $|\Psi_{-1}^k\rangle$  obey different branches of the time evolution process, respectively, leading to that they may be different in the course of the QUANS DAM process even if their corresponding initial states are taken as the same one. Consequently there is the quantum-state difference  $\rho_{12}(k) = \langle \Psi_{+1}^k | \Psi_{-1}^k \rangle$  between the candidate states  $|\Psi_{+1}^k\rangle$  and  $|\Psi_{-1}^k\rangle$  and more importantly the quantum-state difference is varying during the QUANS DAM process, although the QUANS DAM process is unitary.

Owing to these  $QM$  unitary operators  $\{U_k\}$  the original Hilbert subspace of the quantum system in which the basic  $IC$  unitary operator of (2.9) works may be changed during the QUANS DAM process of (3.9). Correspondingly the original reduction subspace of the unstructured search space, where the basic  $IC$  unitary operator works, changes too. But the dimensional size of the reduction subspace spanned by  $\{|\Psi_{+1}^k\rangle, |\Psi_{-1}^k\rangle\}$  of (3.10) during the QUANS DAM process

is always equal to the one of the original reduction subspace except for the special case that  $|\Psi_{+1}^k\rangle$  is the same as  $|\Psi_{-1}^k\rangle$ .

If the reduction subspace in which a general QUANSDAM (or UNIDYSLOCK) process works is  $\mathbb{N}$ -dimensional, then the unitary time evolution process of the QUANSDAM (or UNIDYSLOCK) process owns  $\mathbb{N}$  different branches. Only one of these  $\mathbb{N}$  different branches is quantum-physical and it describes the time evolution process of the candidate state that appears in the quantum system, while the remaining  $\mathbb{N} - 1$  different branches each are mathematical and they describe the time evolution processes of the remaining  $\mathbb{N} - 1$  candidate states of the math Hilbert space of the search problem, respectively. Obviously, the reduction subspace may change during the QUANSDAM (or UNIDYSLOCK) process. It is spanned by these  $\mathbb{N}$  candidate states. Its dimensional size is always equal to  $\mathbb{N}$ , i.e., the one of the original reduction subspace except for the special cases. Generally, these  $\mathbb{N}$  candidate states can be distinguished from each other by the quantum measurement (the usual one or the POVM one) at the end of the QUANSDAM process, when they are orthogonal to each other or approximately orthogonal to each other.

The *QSD* amplification ability for the QUANSDAM process of (3.9) is just the quantum-state-difference ( $\rho_{12}(k)$ ) varying speed. The latter is the characteristic physical quantity of a QUANSDAM (or UNIDYSLOCK) process. Here  $\rho_{12}(k)$  is dependent on the number  $k$  of the basic *IC* unitary operators on the left-hand side of (3.9). It is known from (3.10) that the initial  $\rho_{12}(0) = 1$  because  $|\Psi_{+1}^0\rangle = |\Psi_{-1}^0\rangle = U_0|\Psi_0\rangle$ . Since  $|\rho_{12}(k)| = |\langle\Psi_{+1}^k|\Psi_{-1}^k\rangle| \leq 1$  for  $1 \leq k \leq K$ , on average the varying rate of  $\rho_{12}(k)$  is  $(|\rho_{12}(k)| - |\rho_{12}(0)|)/k$  per basic *IC* unitary operator  $U_\lambda^{ic}(\theta_m, a_m^s)$ . If at the end of the QUANSDAM process both the states  $|\Psi_{+1}^K\rangle$  and  $|\Psi_{-1}^K\rangle$  are orthogonal to one another, then  $\rho_{12}(K) = 0$ . In this case on average the *QSD* varying rate over the whole QUANSDAM process is given by  $-1/K$  per basic *IC* unitary operator. Here one uses only the number of the basic *IC* unitary operators to calculate the average *QSD* varying rate without taking into account the number of the *QM* unitary operators  $\{U_k\}$  in the QUANSDAM process. The main reason for this is that any *QM* unitary operator does not change the quantum-state difference. Since the difference  $|\rho_{12}(k)| - |\rho_{12}(0)|$  is never more than one in magnitude, in a usual case (i.e.,  $||\rho_{12}(k)| - |\rho_{12}(0)|| \approx 1$  with  $k \approx K$ ) the average *QSD* varying rate for the QUANSDAM process is mainly determined by the total number ( $K$ ) of the basic *IC* unitary operators. As shown above, the average *QSD* varying rate is inversely proportional to the total number  $K$ . Then this means that the smaller the total number  $K$ , the larger the average *QSD* amplification ability of the QUANSDAM process.

From the point of view of quantum-computing complexity the complexity of a QUANSDAM process is determined not only by the total number of the basic *IC* unitary operators but also by the total number of the *QM* unitary operators. However, in a usual case the influence of the *QM* unitary operators on the complexity could not be greater than that one of the basic *IC* unitary operators. Then in such case the average *QSD* amplification ability (or the total number of the basic *IC* unitary operators alone) may be conveniently used to



measure approximately the complexity of the QUANSDAM process.

The  $QSD$  amplification ability of a QUANSDAM process could be strongly dependent upon the  $QM$  unitary operators. While a  $QM$  unitary operator itself can not change the quantum-state difference, it is able to promote (or speed up) the  $IC$  unitary operator to change greatly the quantum-state difference, that is, it can enhance the  $IC$  unitary operator to amplify greatly the  $QSD$  varying speed in a QUANSDAM process. Undoubtedly the average  $QSD$  varying rate is convenient to measure the average  $QSD$  amplification ability of a QUANSDAM process as a whole. However, it can not measure exactly the effect of each  $QM$  unitary operator on the  $QSD$  amplification ability of the QUANSDAM process. It may be more exact to employ the following  $QSD$  varying rate:

$$\Delta\rho_{12}(k+1) = |\rho_{12}(k+1)| - |\rho_{12}(k)| \quad (3.11)$$

to measure the effect of each  $QM$  unitary operator ( $U_k$ ) on the  $QSD$  amplification ability of the QUANSDAM process. The physical meaning of  $\Delta\rho_{12}(k+1)$  is that the  $(k+1)$ -th  $IC$  unitary operator in the QUANSDAM process alone changes the quantum-state difference when it acts on the quantum system, and the changing value is given by  $\Delta\rho_{12}(k+1)$ . The reason why  $\Delta\rho_{12}(k+1)$  can be used to measure exactly the effect of the single  $QM$  unitary operator  $U_k$  on the  $QSD$  amplification ability is that there is a difference for  $\Delta\rho_{12}(k+1)$  between there is and there is not the  $QM$  unitary operator  $U_k$  in the QUANSDAM process. In technique it could be more convenient to use  $\Delta\rho_{12}^2(k+1) = |\rho_{12}(k+1)|^2 - |\rho_{12}(k)|^2$  rather than  $\Delta\rho_{12}(k+1)$ . Beside the  $QSD$  varying rate  $\Delta\rho_{12}(k+1)$  one also may use the average  $QSD$  varying rate over the  $L$  basic  $IC$  unitary operators ( $L \ll K$ ) to measure the average effect of the  $L$  number of the  $QM$  unitary operators on the  $QSD$  amplification ability. Below it is described simply how a  $QM$  unitary operator could be appropriate to enhance the  $QSD$  amplification ability of a QUANSDAM process.

First of all, consider the special case that all the  $QM$  unitary operators  $\{U_k\}$  are the unit operators in the QUANSDAM process of (3.9). Then from (3.9) one obtains

$$(U_\lambda^{ic}(\theta_m, a_m^s))^K |\Psi_0\rangle = \begin{cases} |\Psi_{+1}^K\rangle, & \text{if } a_m^s = +1 \\ |\Psi_{-1}^K\rangle, & \text{if } a_m^s = -1 \end{cases} \quad (3.12a)$$

This QUANSDAM process (without the initial state) consists of the  $K$  basic  $IC$  unitary operators of (2.9) alone. There is not any contribution made by any  $QM$  unitary operator to its  $QSD$  amplification ability. Therefore, the QUANSDAM process of (3.12a) may be considered as the reference QUANSDAM process. If a QUANSDAM process has the  $QSD$  amplification ability stronger than the one of (3.12a), then its  $QM$  unitary operators make a positive contribution to the  $QSD$  amplification ability. In contrast, if a QUANSDAM process has the  $QSD$  amplification ability weaker than the one of (3.12a), then its  $QM$  unitary operators are inappropriate for the QUANSDAM process.

By using (2.9) the QUANSDAM process of (3.12a) (without the initial state) is explicitly written as  $(U_\lambda^{ic}(\theta_m, a_m^s))^K = \exp(-ia_m^s K \theta_m I_{m\lambda})$ . Note that the

angle  $\theta_m \propto 1/2^n$  is exponentially small. Then the number  $K$  of the basic *IC* unitary operators must be sufficiently large so that the solution information ( $a_m^s$ ) can be obtained by the QUANSDAM process. As a typical example, the QUANSDAM process with  $\lambda = x$  is applied to the initial state  $|\Psi_0\rangle = |0\rangle$  and just like the simple QUANSDAM process of (3.4) one obtains

$$(U_x^{ic}(\theta_m, a_m^s))^K |0\rangle = \cos\left(\frac{K}{2}\theta_m\right) |0\rangle - ia_m^s \sin\left(\frac{K}{2}\theta_m\right) |1\rangle. \quad (3.12b)$$

This indicates that the two candidate states at the end of the QUANSDAM process of (3.12a) are given by  $|\Psi_{\pm 1}^K\rangle = \cos\left(\frac{K}{2}\theta_m\right) |0\rangle \mp i \sin\left(\frac{K}{2}\theta_m\right) |1\rangle$ . Then the scalar product between the two states is given by  $\rho_{12}(K) = \langle \Psi_{+1}^K | \Psi_{-1}^K \rangle = \cos(K\theta_m)$ . When  $\cos(K\theta_m) = 0$  or when the number  $K$  is given by  $K = \frac{1}{2}\pi/|\theta_m|$ , the scalar product  $\rho_{12}(K) = 0$ , indicating that both the candidate states  $|\Psi_{+1}^K\rangle$  and  $|\Psi_{-1}^K\rangle$  are orthogonal to each other. Now by the usual quantum measurement [17, 14] one is able to determine the unit number ( $a_m^s$ ) unambiguously. Since the angle  $\theta_m$  is exponentially small, the number  $K$  has to be exponentially large. This shows that the QUANSDAM process of (3.12) that consists of only the basic *IC* unitary operators is an inefficient process to amplify the quantum-state difference.

The *QSD* varying rates for the QUANSDAM process of (3.12b) are calculated below. The average *QSD* varying rate over the whole process is given by  $-1/K = -2|\theta_m|/\pi$ . It is exponentially small in magnitude, indicating that the QUANSDAM process is inefficient. It is easy to find that the scalar product  $\rho_{12}(k) = \langle \Psi_{+1}^k | \Psi_{-1}^k \rangle = \cos(k\theta_m)$  for  $0 \leq k \leq K$ . Then the average *QSD* varying rate of  $\rho_{12}(k)$  over the first  $k$  basic *IC* unitary operators is

$$(|\rho_{12}(k)| - |\rho_{12}(0)|)/k = (\cos(k\theta_m) - 1)/k \approx -\frac{1}{4}\pi |\theta_m| k/K \quad (3.13)$$

where  $k|\theta_m| \ll 1$ . By using the formula (3.11) it can turn out that the *QSD* varying rate  $\Delta\rho_{12}(k+1)$  ( $0 \leq k \leq K-1$ ) is given by

$$\Delta\rho_{12}(k+1) = -2 \sin\left(\left(k + \frac{1}{2}\right)\theta_m\right) \sin\left(\frac{1}{2}\theta_m\right) \quad (3.14)$$

where  $(k+1)|\theta_m| \leq \pi/2$ . Note that  $\theta_m$  is exponentially small. Then one has  $\sin\left(\frac{1}{2}|\theta_m|\right) \approx \frac{1}{2}|\theta_m| = \frac{1}{4}\pi/K$ . When  $k|\theta_m| \ll 1$ , one obtains from (3.14) that

$$\Delta\rho_{12}(k+1) \approx -\frac{1}{4}\pi |\theta_m| (2k+1)/K. \quad (3.15)$$

Both  $(|\rho_{12}(k)| - |\rho_{12}(0)|)/k < 0$  ( $1 \leq k \leq K$ ) and  $\Delta\rho_{12}(k+1) < 0$  ( $0 \leq k \leq K-1$ ) are consistent with the fact that the process of (3.12b) is a QUANSDAM process.

As shown in (3.15), when  $k|\theta_m| \ll 1$ , the *QSD* varying rate  $\Delta\rho_{12}(k+1)$  is linearly dependent on the value  $k|\theta_m|$  or the number ( $k$ ) of the basic *IC* unitary operators in the QUANSDAM process. It can prove that this conclusion

also holds for the QUANSDAM process of (3.12a) with an arbitrary initial state  $|\Psi_0\rangle$ . As shown in (3.13), the average  $QSD$  varying rate of  $\rho_{12}(k)$  is linearly dependent on the value  $k|\theta_m|$  or the number  $k$  too. Clearly, the  $QSD$  varying rate  $\Delta\rho_{12}(k+1)$  for the QUANSDAM process of (3.12) generally decreases linearly (or it increases linearly in magnitude) as the value  $k|\theta_m|$  or the number  $k$  of the basic  $IC$  unitary operators. Here the condition  $k|\theta_m| \ll 1$  is quite general, because one may set  $K|\theta_m| = O(1/p(n)) \ll 1$  and  $0 \leq k < K$ , here  $p(n)$  is a polynomial in the qubit number  $n$  corresponding to the  $2^n$ -dimensional unstructured search space. However, if  $K|\theta_m| = O(1/p(n)) \ll 1$ , then  $\rho_{12}(K) = \cos(K\theta_m) \neq 0$ , indicating that both the candidate states  $|\Psi_{+1}^K\rangle$  and  $|\Psi_{-1}^K\rangle$  are not orthogonal to each other at the end of the QUANSDAM process of (3.12). In this case in technique one is still able to distinguish the two non-orthogonal states  $|\Psi_{+1}^K\rangle$  and  $|\Psi_{-1}^K\rangle$  from each other and obtain the solution information ( $a_m^s$ ) by the quantum measurement with unambiguous state distinguishability [28], but the maximum successful probability is  $P = 1 - |\rho_{12}(K)|$ .

There is not any  $QM$  unitary operator in the QUANSDAM process of (3.12). The characteristic feature for such QUANSDAM process is that the  $QSD$  varying rate  $\Delta\rho_{12}(k+1)$  decreases linearly as the value  $k|\theta_m|$  or the number ( $k$ ) of the basic  $IC$  unitary operators in the QUANSDAM process. The QUANSDAM process may be considered as the reference one for investigating the influence of a  $QM$  unitary operator on the  $QSD$  amplification ability of a general QUANSDAM process.

The QUANSDAM process of (3.12) may be called a linear QUANSDAM process due to that its  $QSD$  varying rate  $\Delta\rho_{12}(k+1)$  decreases linearly as the number ( $k$ ) of the basic  $IC$  unitary operators. Just like the linear QUANSDAM process one may define other QUANSDAM processes. A QUANSDAM process whose  $\Delta\rho_{12}(k+1)$  decreases in a square (or quadratic) form as the number  $k$  may be called a square QUANSDAM process. Of course, here  $\Delta\rho_{12}(k+1) \propto |\theta_m|/K$  too. If the  $QSD$  varying rate  $\Delta\rho_{12}(k+1)$  of a QUANSDAM process decreases in the form of a polynomial or an exponential function as the number  $k$ , then the QUANSDAM process may be called a polynomial or an exponential QUANSDAM process. As shown above, there is not any  $QM$  unitary operator in the linear QUANSDAM process of (3.12). Then a super-linear QUANSDAM process exists only when it contains the  $QM$  unitary operators. Note that a  $QM$  unitary operator itself can not change the quantum-state difference. These imply that a super-linear QUANSDAM process must be involved in the interaction between the  $IC$  unitary operators and the  $QM$  unitary operators in the QUANSDAM process.

Here it must be pointed out that because of the parameter  $\theta_m \propto 1/2^n$  a square QUANSDAM process above does not mean that the extraction of the solution information ( $a_m^s$ ) can be realized in a square speedup by the QUANSDAM process. Actually it can prove that a cubic QUANSDAM process above is able to extract the solution information in a square speedup. However, if one employs the  $QSD$  varying rate  $\Delta\rho_{12}(k+1)/k$  or  $\Delta\rho_{12}^2(k+1)/k$  to measure a QUANSDAM process, then a square QUANSDAM process measured in this way is able to obtain the solution information in a square speedup and hence

it is correctly correspondent to the amplitude amplification of a conventional quantum search algorithm. In order to retain this correspondence relation, whenever one compares a QUANSDAM process with the amplitude amplification of a conventional quantum search algorithm in this paper, it implies that the QUANSDAM process is measured by the  $QSD$  varying rate  $\Delta\rho_{12}(k+1)/k$  (or  $\Delta\rho_{12}^2(k+1)/k$ ). Here, if  $k=0$ , then  $\Delta\rho_{12}(k+1)/k$  is set to  $\Delta\rho_{12}(k+1)$ .

In principle an  $IC$  unitary operator like  $U_\lambda^{ic}(\theta_m, a_m^s)$  of (2.9) acting on a quantum state with the solution information ( $a_m^s$ ) may become a QUANSDAM process or a UNIDYSLOCK process. As known from (3.9), one has  $|\Psi_{a_m^s}^k\rangle = QUANSDAM(k, a_m^s)|\Psi_0\rangle$  with  $0 \leq k \leq K$ . Suppose now that  $|\Phi_{a_m^s}^{k+1}\rangle = U_\lambda^{ic}(\theta_m, a_m^s)|\Psi_{a_m^s}^k\rangle$ . Then  $|\Psi_{a_m^s}^{k+1}\rangle = U_{k+1}|\Phi_{a_m^s}^{k+1}\rangle$ . It can be found that the quantum-state difference  $\rho_{12}(k) = \langle\Psi_{+1}^k|\Psi_{-1}^k\rangle = \langle\Phi_{+1}^k|\Phi_{-1}^k\rangle$ , indicating that the  $QM$  unitary operator  $U_k$  does not affect  $\rho_{12}(k)$ . Though  $U_k$  does not affect  $\rho_{12}(k)$ , it can affect  $\rho_{12}(k+1)$ . If now  $|\rho_{12}(k+1)| > |\rho_{12}(k)|$ , then  $|\Phi_{a_m^s}^{k+1}\rangle = U_\lambda^{ic}(\theta_m, a_m^s)U_k|\Phi_{a_m^s}^k\rangle$  is a UNIDYSLOCK process. If  $|\rho_{12}(k+1)| < |\rho_{12}(k)|$ , then  $|\Phi_{a_m^s}^{k+1}\rangle = U_\lambda^{ic}(\theta_m, a_m^s)U_k|\Phi_{a_m^s}^k\rangle$  is a QUANSDAM process. Here the  $QM$  unitary operator  $U_k$  plays a crucial role in that it can increase the varying speed and adjust the varying direction of quantum-state difference in a deterministic form for the next  $IC$  unitary operator  $U_\lambda^{ic}(\theta_m, a_m^s)$ .

Consider the extended version of the reference QUANSDAM process of (3.12):

$$|\tilde{\Psi}_{a_m^s}^k\rangle = (U_\lambda^{ic}(\theta_m, a_m^s))^k |\tilde{\Psi}_{a_m^s}\rangle \quad (3.16)$$

where the initial state  $|\tilde{\Psi}_{a_m^s}\rangle$  is allowed to carry the solution information ( $a_m^s$ ). Unlike the QUANSDAM process of (3.12), the process of (3.16) may be a QUANSDAM process or a UNIDYSLOCK process. In either case it can prove that when  $k|\theta_m| \ll 1$ , for an arbitrary initial state  $|\tilde{\Psi}_{a_m^s}\rangle$  the  $QSD$  varying rate  $\Delta\rho_{12}^2(k+1)$  is linearly dependent on the value  $k|\theta_m|$  or the number ( $k$ ) of the basic  $IC$  unitary operators in (3.16). Thus, this process reproduces the characteristic feature of the reference QUANSDAM process of (3.12).

A  $QM$  unitary operator should make a positive contribution to the  $QSD$  amplification ability for its own QUANSDAM process. Here the so-called positive contribution means that, promoted by the  $QM$  unitary operators, the QUANSDAM process has a stronger  $QSD$  amplification ability than the linear one of the reference QUANSDAM process of (3.12). However, a linear or super-linear QUANSDAM process could not be useful. A useful QUANSDAM process should have the  $QSD$  varying rate  $\Delta\rho_{12}(k+1)/k \propto k^2$  or more in magnitude, so that it is able to extract the solution information in a square speedup at least. Now a  $QM$  unitary operator (e.g.,  $U_k$ ) of a QUANSDAM process is said to be appropriate if it is able to make the  $QSD$  varying rate  $\Delta\rho_{12}(k+1)/k$  (or  $\Delta\rho_{12}^2(k+1)/k$ ) in magnitude exceeding  $O(p_2(k))$ , where  $p_2(k)$  is a quadratic polynomial in the number  $k$ . A QUANSDAM process is obviously able to extract the solution information in a super-square speedup if every  $QM$  unitary operator of it is appropriate.

Suppose now that the  $QM$  unitary operators  $\{U_k\}$  with  $0 \leq k < K$  are

appropriate in the QUANSDAM process of (3.9). Then one may say that the quantum state  $|\Psi_{a_m^s}^k\rangle = U_k|\Phi_{a_m^s}^k\rangle = \text{QUANSDAM}(k, a_m^s)|\Psi_0\rangle$  is appropriate. By applying the basic *IC* unitary operator to the appropriate quantum state  $|\Psi_{a_m^s}^k\rangle$  one obtains  $|\Phi_{a_m^s}^{k+1}\rangle = U_\lambda^{ic}(\theta_m, a_m^s)|\Psi_{a_m^s}^k\rangle = U_\lambda^{ic}(\theta_m, a_m^s)U_k|\Phi_{a_m^s}^k\rangle$ , and both  $\rho_{12}(k+1) = \langle\Phi_{+1}^{k+1}|\Phi_{-1}^{k+1}\rangle$  and  $\rho_{12}(k) = \langle\Phi_{+1}^k|\Phi_{-1}^k\rangle$  lead to that the *QSD* varying rate  $\Delta\rho_{12}^2(k+1)/k$  can exceed  $O(p_2(k))$  in magnitude. In contrast, if  $U_k$  is not appropriate, then the state  $|\Psi_{a_m^s}^k\rangle = U_k|\Phi_{a_m^s}^k\rangle$  is not appropriate and hence  $\Delta\rho_{12}^2(k+1)/k$  can not exceed  $O(p_2(k))$  in magnitude. If there is not the  $k$ -th *QM* unitary operator  $U_k$ , then one has  $|\Phi_{a_m^s}^{k+1}\rangle = U_\lambda^{ic}(\theta_m, a_m^s)|\Phi_{a_m^s}^k\rangle = (U_\lambda^{ic}(\theta_m, a_m^s))^2|\Psi_{a_m^s}^{k-1}\rangle$ . As shown in (3.16), in this case the *QSD* varying rate  $\Delta\rho_{12}^2(k+1)$  has the characteristic feature of the reference QUANSDAM process of (3.12) or (3.16).

It is important how to find the appropriate *QM* unitary operators so that one can construct a useful QUANSDAM process in a quantum system. In the next section this research topic is further discussed.

There are two quite different types of QUANSDAM processes. One of which is the amplitude-based QUANSDAM processes and another is the phase-based QUANSDAM processes. The former is relatively simple and similar to the amplitude amplification of a conventional quantum search algorithm and they usually work in an  $n$ -qubit quantum system, while the latter is quite complex and they work generally in a time- and space-dependent quantum system. Below an amplitude-based QUANSDAM process is introduced simply. A phase-based QUANSDAM process is closely related to the so-called information-carrying unitary propagator. They are described in the next section.

The fashion of an amplitude-based QUANSDAM process is that the same basic *IC* unitary operator and the *QM* unitary operator appear alternatively in the QUANSDAM process. An amplitude-based QUANSDAM process is quite similar to the amplitude amplification of a conventional quantum search algorithm. Moreover, there is the corresponding relation between them. Actually, the simple QUANSDAM process of (3.4) and the one of (3.12) may be considered as the special amplitude-based QUANSDAM processes. The QUANSDAM process of (3.9) is a general amplitude-based QUANSDAM process. Usually all the *QM* unitary operators  $\{U_k\}$  in (3.9) are taken as the same one and doing so could greatly simplify the exact calculation of the QUANSDAM process. Though an amplitude-based QUANSDAM process could be calculated exactly, it could not capture exactly the essential aspect of a general QUANSDAM process.

As an example, below analyze the amplitude-based QUANSDAM process of (3.12b). It is known from (3.12b) that the state  $|1\rangle$  carries the solution information  $(a_m^s)$ . Here for convenience the state  $a_m^s|1\rangle$  in (3.12b) is called the information-carrying state (or the *IC* state briefly). In (3.12b) the *IC* state  $a_m^s|1\rangle$  has an amplitude of  $-i \sin \frac{K}{2}\theta_m$ . The absolute amplitude value  $|\sin \frac{K}{2}\theta_m|$  may act as the measure of the quantum-state difference between the final two candidate states  $\{|\Psi_{\pm 1}^K\rangle\}$  of the QUANSDAM process of (3.12b). When the number  $K$  increases from 0 to its maximum value  $K = \frac{1}{2}\pi/|\theta_m|$ , the amplitude

value grows from the initial value 0 to the final value  $1/\sqrt{2}$ , leading to that both the final states  $\{|\Psi_{\pm 1}^K\rangle\}$  evolve from the identical initial states to the orthogonal states.

For a general amplitude-based QUANSDAM process of (3.9) the situation may not be so simple as the above one of (3.12b). Suppose that at the end of the QUANSDAM process of (3.9) the state may be formally written as

$$\Psi_{a_m^s}(\theta_m, K) = \Psi_a(\theta_m, K) + a_m^s \Psi_b(\theta_m, K) \quad (3.17)$$

where  $a_m^s \Psi_b(\theta_m, K)$  is the *IC* state. The final state  $\Psi_{a_m^s}(\theta_m, K)$  is normalized, but the *IC* state may not. Then the amplitude of the *IC* state is just  $\|\Psi_b(\theta_m, K)\|$  in magnitude. The normalization condition for the final state  $\Psi_{a_m^s}(\theta_m, K)$  is given by  $\|\Psi_{a_m^s}(\theta_m, K)\|^2 = 1$ . By using (3.17) it is reduced to the form

$$\|\Psi_a(\theta_m, K)\|^2 + \|\Psi_b(\theta_m, K)\|^2 = 1 \quad (3.18a)$$

$$\Psi_a^+(\theta_m, K) \Psi_b(\theta_m, K) + \Psi_b^+(\theta_m, K) \Psi_a(\theta_m, K) = 0 \quad (3.18b)$$

On the other hand, the orthogonal condition for the final two candidate states  $|\Psi_{+1}^K\rangle$  and  $|\Psi_{-1}^K\rangle$  in (3.9) is given by  $\langle \Psi_{+1}^K | \Psi_{-1}^K \rangle = 0$ . Now from (3.17) one obtains  $|\Psi_{+1}^K\rangle = \Psi_a(\theta_m, K) + \Psi_b(\theta_m, K)$  and  $|\Psi_{-1}^K\rangle = \Psi_a(\theta_m, K) - \Psi_b(\theta_m, K)$ . Thus, the orthogonal condition may be reduced to the form

$$\langle \Psi_{+1}^K | \Psi_{-1}^K \rangle = 1 - 2 \|\Psi_b(\theta_m, K)\|^2 - 2 \Psi_a^+(\theta_m, K) \Psi_b(\theta_m, K) = 0 \quad (3.19)$$

where the normalization condition (3.18) is already used. It is clearly dependent on not only the absolute amplitude value  $\|\Psi_b(\theta_m, K)\|$  but also the scalar product  $\Psi_a^+(\theta_m, K) \Psi_b(\theta_m, K)$ . The orthogonal condition (3.19) is universal even for a general QUANSDAM process. It could be helpful to determine the minimum number  $K$  of a general amplitude-based QUANSDAM process of (3.9).

#### 4. The phase-based QUANSDAM process and the information-carrying unitary propagators

It was first proposed in Ref. [7] that a UNIDYSLOCK process could be constructed in a single-atom quantum system which is a time- and space-dependent quantum system. The basic theoretical method employed to construct a UNIDYSLOCK process and its inverse in a single-atom system is the unitary manipulation of a single atom in time and space. Since then, the unitary manipulation of a single-atom system in time and space has been insistently studied in theory in the past decade [7, 31, 19, 15]. The main purpose for these works is to realize an exponential QUANSDAM (or UNIDYSLOCK) process in a single-atom system and investigate the quantum-computing speedup mechanism for the unitary quantum dynamics. The work in this section continues along the same research direction and toward the same goal. These works and the present work lay down the basis for the coming exponential QUANSDAM process in a single-atom system.

Since they were first chosen in Ref. [7], Gaussian wave-packet states and quadratic unitary propagators [7, 31, 19, 15] have been used as the basic quantum states and the basic unitary propagators in the unitary manipulation of a

single atom in time and space, respectively. That a Gaussian wave-packet state was used in the unitary manipulation in the early time [7] is closely related to the construction of a spatially selective and internal-state selective triggering pulse, with which the author attempted to realize a reversible and unitary halting protocol in a single-atom system (i.e., the so-called halting-qubit atom), and also related to the realization of a UNIDYSLOCK process. The spatially selective and internal-state selective triggering pulse [7, 31, 15] needs a narrow wave-packet motional state of the single-atom system in one-dimensional coordinate space. Here only the (internal-state-dependence) spatially selective excitation of the triggering pulse requires the atomic wave-packet motional state to have a narrow spatial wave-packet spread. It is therefore important to choose a narrower atomic wave-packet motional state to realize the spatially selective and internal-state selective triggering pulse and study a UNIDYSLOCK (or QUANS DAM) process (See also Appendix A for the relation between the spatially selective and internal-state selective triggering pulse and the phase-based QUANS DAM process).

Initially the present author chose a Gaussian (shaped) wave-packet motional state mainly based on the author's experience of experimental research on the NMR application [32] of the Gaussian shaped pulses. It was shown in the selective excitation experiments [32] that a Gaussian shaped pulse, which is a radio-frequency electromagnetic wave pulse [23], has a narrower waveform width ( $\Delta t$ ) in the time domain (i.e., a shorter pulse width) than other simple or complex (examined) shaped pulses under the same excitation bandwidth ( $\Delta\omega = \Delta E/\hbar$ ) in the frequency (or energy) domain. Thus, it was thought by the author that a Gaussian (shaped) wave-packet motional state of the single-atom system should have a narrower spatial wave-packet spread ( $\Delta x$ ) than other (shaped) wave-packet motional states under the same conditions which include the same momentum spread ( $\Delta p$ ). That a Gaussian wave-packet motional state was chosen as the basic quantum state to study a spatially selective and internal-state selective triggering pulse and a UNIDYSLOCK process [7] is merely a correct and important starting point. The theoretical basis and the experimental basis for a Gaussian wave-packet state to be the basic quantum state and a quadratic unitary propagator to be the basic unitary propagator mainly come from the three aspects [15]. On the first aspect it has been shown in quantum physics that a Gaussian wave-packet state can keep its Gaussian shape unchanged under action of a quadratic unitary propagator (See, for example, Refs. [21, 24]), while a quadratic unitary propagator can be calculated exactly and conveniently by the Feynman path integral [18]. On the second aspect in quantum physics the time evolution process of a Gaussian wave-packet state can be calculated exactly and conveniently under action of a quadratic unitary propagator. Here a Gaussian wave-packet state may be a standard one or a non-standard one (See Ref. [15]). On the third aspect in the coherent manipulation and control of a single-atom system (mainly in experiment) a Gaussian wave-packet motional state has been studied (See, for example, Refs. [34]); and a single atom in an external harmonic potential field can be perfectly manipulated or controlled experimentally (See Ref. [30b]). Gaussian wave-packet motional states [7, 31, 19, 15] continue

to play a key and basic role in realizing the coming exponential QUANSDAM process in a single-atom system [36].

A single atom is one of the simplest quantum systems that involve the center-of-mass (COM) motion (or external motion) and the internal motion of quantum system at the same time. It may have four degrees of freedom that can be unitarily manipulated independently, one of which is the internal motion degree of freedom and the others are the three independent COM motion degrees of freedom in three-dimensional coordinate space. In particular, the atomic COM motion is allowed to be manipulated unitarily in an arbitrary manner. Moreover, both the atomic COM motion and internal motion can interact with each other, when the external electromagnetic field is appropriately applied to the single-atom system. Therefore, a complete description for a single-atom system must consider both the atomic COM motion and internal motion as well as the interaction between the two motions in the study of the unitary manipulation of a single-atom system in time and space. On the other hand, unlike a single atom motioning in coordinate space each qubit of an  $n$ -qubit quantum system need not consider explicitly its COM motion in quantum computation, even if the COM motion (or external motion) could exist, since a quantum computation in an  $n$ -qubit quantum system usually does not explicitly depend on the external motion of each qubit of the quantum system. Therefore, in a usual case only the internal motions of the qubits and the interaction between the internal motions in the  $n$ -qubit quantum system need to be considered explicitly in quantum computation.

Unlike a composite quantum system, a single quantum system such as a single-atom system does not own the fundamental quantum-computing resource. In the last section it is shown in theory that a single quantum system could be appropriate to realize an exponential QUANSDAM (or UNIDYSLOCK) process, when the basic building block of the QUANSDAM (or UNIDYSLOCK) process, i.e., the basic  $IC$  unitary operator prepared by the search-space dynamical reduction works in a smallest reduction subspace. Because the basic  $IC$  unitary operator works in a smallest reduction subspace, the effect of the symmetric structure of the Hilbert space of the quantum system could become secondary on the QUANSDAM (or UNIDYSLOCK) process which is performed in the quantum system. Consequently one could choose an appropriate single quantum system such as a single-atom system to realize an exponential QUANSDAM (or UNIDYSLOCK) process. A single quantum system is simpler and easier to manipulate and control unitarily.

As known in the section 2, the basic  $IC$  unitary operator of (2.9) works in the smallest reduction subspace of the math Hilbert space of the search problem, i.e., a two-dimensional reduction subspace. Then in performance a QUANSDAM (or UNIDYSLOCK) process whose basic building block is the basic  $IC$  unitary operator could not be affected significantly by the symmetric structure of the Hilbert space of the quantum system that performs the QUANSDAM (or UNIDYSLOCK) process. Therefore, such QUANSDAM process even with an exponential  $QSD$  amplification ability could be realized in a single-atom system. Below in principle it is described how an exponential QUANSDAM



process could be realized (mainly but no limited to) in a single-atom system.

It is known from the basic *IC* unitary operator of (2.9) that the  $m$ -th spin (or qubit) of the  $n$ -qubit quantum system carries the information ( $a_m^s$ ) of the  $m$ -th component state of the solution state. When a QUANS DAM (or UNIDYSLOCK) process is constructed in a single-atom system, as one key step of the construction the solution information ( $a_m^s$ ) needs first to be transferred in a unitary form from the  $m$ -th spin (or  $m$ -th qubit) of the  $n$ -qubit quantum system to the single-atom system and the important is that the solution information is transferred to the COM motion (degrees of freedom) of the single-atom system. The best way to realizing this information transfer is perhaps that the solution information is first transferred from the  $m$ -th spin to the atomic internal motion and then it is further transferred from the atomic internal motion to the atomic COM motion. When the solution information is transferred from the atomic internal motion to the atomic COM motion, one needs to employ the interaction between the internal motion and the COM motion of the single-atom system to realize such information transfer. Such an interaction tends to be internal-state selective (or dependent) in a single-atom system. It can be easily realized in a single-atom system. It is well known that the interaction between the internal and COM motions of a single atom has been used extensively in the atomic laser light cooling [29], the atomic quantum coherence interference [29a], the atomic quantum-computing implementation [30], and the coherent manipulation and control of a single-atom system [30b].

The interaction between the internal motion and the COM motion of a single-atom system also plays an important role in realizing conveniently the selective excitation of the COM energy eigenstates in the single-atom system. When a single atom is in an external harmonic potential field, the total atomic motion consists of the atomic internal motion and external COM harmonic motion. For simplicity here consider that the single-atom system is in one-dimensional COM harmonic motion. On the one hand, according to the standard one-dimensional harmonic-oscillator quantum theory [14] the atomic COM harmonic motion is discrete and has equal-spacing energy levels. Such an equal-spacing energy level structure tends to be an obstacle to realizing conveniently the selective excitation of the COM energy states of the single-atom system. On the other hand, the internal energy level structure of a single atom is rich. The atomic internal energy levels are generally discrete and not equal-spacing. For a single atom in an external harmonic potential field, its total energy consists of the atomic internal energy and external energy of the COM harmonic motion. The atomic energy level structure therefore is quite different from the simple equal-spacing energy level structure. Then the selective excitation of the atomic COM energy states may be performed more conveniently with the help of the atomic internal energy levels. Such a selective excitation is involved in both the atomic internal motion and external COM harmonic motion. The interaction between the atomic internal motion and external COM harmonic motion therefore is necessary to realize the selective excitation. On the basis of the interaction these well-established selective excitation techniques could be used to realize it, which include the selective excitation techniques in NMR spectroscopy [23] and

the STIRAP technique [33] and so on.

After the solution information ( $a_m^s$ ) is transferred to the internal motion of the single-atom system, the basic  $IC$  unitary operator of (2.9) originally working in the two-dimensional subspace of the  $m$ -th qubit of the  $n$ -qubit quantum system is changed to the one working in a two-dimensional internal-state subspace of the Hilbert space of the atomic internal motion. For convenience hereafter this new basic  $IC$  unitary operator of the single-atom system is still denoted as the same form as (2.9), i.e.,  $U_\lambda^{ic}(\theta_m, a_m^s) = \exp(-ia_m^s \theta_m I_{m\lambda})$ , where  $I_{m\lambda}$  is a spin-1/2 (or pseudospin-1/2) operator of the atomic internal motion. Now by starting from this basic  $IC$  unitary operator of the single-atom system one may construct the desired QUANSDAM (or UNIDYSLOCK) process in the single-atom system.

As pointed out in the preceding section, there are two different types of QUANSDAM (or UNIDYSLOCK) processes. They are the amplitude-based and the phase-based QUANSDAM (or UNIDYSLOCK) processes, respectively. A phase-based QUANSDAM process may be quite different from an amplitude-based counterpart. It works generally in a time- and space-dependent quantum system such as a single-atom system, while an amplitude-based QUANSDAM process and the amplitude amplification of a conventional quantum search algorithm [3, 13] usually work in an  $n$ -qubit quantum system. As shown in the preceding section, it is relatively simple to construct an amplitude-based QUANSDAM process like (3.9). This is a bottom-to-top method to design a QUANSDAM process. Suppose that in (3.9) the time evolution process for every  $QM$  unitary operator acting on the quantum system can be calculated exactly. Then the amplitude-based QUANSDAM process of (3.9) may be calculated exactly. However, the two final candidate states of (3.9) can be obtained only after the whole QUANSDAM process of (3.9) is calculated exactly. Therefore, it is not easy to find the two candidate orthogonal states at the end of the amplitude-based QUANSDAM process with many  $QM$  unitary operators. In contrast, as shown below, the two candidate orthogonal states at the end of a phase-based QUANSDAM process may be determined easily, but it is an uneasy task how to construct efficiently the phase-based QUANSDAM process itself. This is a top-to-bottom method to design a QUANSDAM process.

Below it is described in detail how a phase-based QUANSDAM process works in a single-atom system. First of all, from the point of view of quantum mechanics a simple introduction is given to the momentum eigenfunctions and their orthogonal relations for a free particle such as a single atom motioning freely in one-dimensional coordinate space. It is well-known in quantum mechanics [14] that a momentum eigenfunction of a free particle may take the box normalization form or the Dirac  $\delta$ -function normalization form. The former is a discrete form, while the latter is a continuous form. Below the box normalization momentum eigenfunction is mainly used. Now consider a momentum eigenfunction of the particle in one-dimensional box with an arbitrary large length  $L$  centered at the origin, here the eigenfunction satisfies the periodic boundary condition. This momentum eigenfunction in the box normalization form may be written

as [14]

$$|\Psi_k\rangle = \frac{1}{\sqrt{L}} \exp(ip_k x/\hbar) \quad (4.1)$$

where the discrete momentum eigenvalue  $p_k = 2\pi\hbar k/L$  with quantum number  $k = 0, \pm 1, \pm 2, \dots$ . It satisfies the orthonormal condition:

$$\langle \Psi_k | \Psi_l \rangle = \delta_{kl} \quad (4.2)$$

where the  $\delta$ -function  $\delta_{kl}$  is given by  $\delta_{kl} = 0$  if  $k \neq l$  and  $\delta_{kl} = 1$  if  $k = l$ . The relation (4.2) shows that both the momentum eigenfunctions  $|\Psi_k\rangle$  and  $|\Psi_l\rangle$  are orthogonal to each other when both the quantum numbers  $k$  and  $l$  are different, i.e.,  $k \neq l$ . Moreover, when the box length  $L \rightarrow \infty$ , any pair of momentum eigenfunctions are orthogonal to each other no matter how small their momentum difference  $p_k - p_l$  is as long as the momentum difference does not equal zero.

Now suppose that an *IC* unitary operator is taken as  $\exp(-ia_m^s p'_0 x/\hbar)$  for a single atom in the box with one-dimensional coordinate space  $(-L/2, L/2)$ . This *IC* unitary operator is called the *IC* unitary momentum-displacement propagator<sup>8</sup>. It is spatially dependent. It is quite different from the basic *IC* unitary operator of (2.9) of the single-atom system. But it is generally hard to prepare the *IC* unitary momentum-displacement propagator by using the basic *IC* unitary operators and the *QM* unitary operators. It may be more direct and reasonable for the single-atom system to prepare the *IC* unitary operator:

$$U_p^{ic}(a_m^s) = \exp(-ia_m^s p'_0 x S_z/\hbar) \quad (4.3)$$

where  $S_z$  is the  $z$ -component spin operator of the atomic internal motion. This *IC* unitary operator may be called the internal-motion-dependent *IC* unitary momentum-displacement propagator. Unlike the *IC* unitary operator  $\exp(-ia_m^s p'_0 x/\hbar)$  it is able to reflect the interaction between the internal motion and the COM motion of the single-atom system. As shown in Appendix A and also in Refs. [15, 31], it could be generated approximately by using the basic *IC* unitary operators and the suitable *QM* unitary operators. However, so far there is not a rigorous mathematical proof to show that the approximately generated *IC* unitary propagator  $U_p^{ic}(a_m^s)$  can lead to an exponential QUANSDAM process.

Just like the basic *IC* unitary operator, the *IC* unitary propagator  $U_p^{ic}(a_m^s)$  also carries the solution information  $(a_m^s)$ . Suppose that the single atom is in the product state  $|\Psi_k\rangle|0\rangle$ , where  $|0\rangle$  and  $|\Psi_k\rangle$  are the atomic internal ground state and COM momentum eigenfunction given by (4.1), respectively. When  $U_p^{ic}(a_m^s)$  acts on the atomic product state  $|\Psi_k\rangle|0\rangle$ , one obtains the QUANSDAM process:

$$U_p^{ic}(a_m^s)|\Psi_k\rangle|0\rangle = \frac{1}{\sqrt{L}} \exp(i(p_k - a_m^s p'_0 m_z)x/\hbar)|0\rangle \quad (4.4a)$$

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<sup>8</sup>That  $\exp(-ip_0 x/\hbar)$  is called the unitary momentum-displacement propagator is due to the unitary transformation  $\exp(ip_0 x/\hbar)p \exp(-ip_0 x/\hbar) = p - p_0$ .

where the eigen-equation  $S_z|0\rangle = m_z|0\rangle$  with the eigenvalue  $m_z \neq 0$  is already used. There are two candidate momentum wavefunctions  $|\psi_{\pm 1}\rangle$  at the end of the QUANS DAM process. These two wavefunctions may be explicitly obtained by rewriting (4.4a) as

$$U_p^{ic}(a_m^s)|\Psi_k\rangle|0\rangle = \begin{cases} |\psi_{+1}\rangle|0\rangle = \frac{1}{\sqrt{L}} \exp(i(p_k - m_z p'_0)x/\hbar)|0\rangle, & \text{if } a_m^s = +1 \\ |\psi_{-1}\rangle|0\rangle = \frac{1}{\sqrt{L}} \exp(i(p_k + m_z p'_0)x/\hbar)|0\rangle, & \text{if } a_m^s = -1 \end{cases} \quad (4.4b)$$

Here the important point is that only one of the two candidate states  $\{|\psi_{+1}\rangle|0\rangle, |\psi_{-1}\rangle|0\rangle\}$  appears in the single-atom system, while another does not. When the box length  $L \rightarrow \infty$ , both the wavefunctions  $\{|\psi_{\pm 1}\rangle\}$  are the momentum eigenfunctions  $\{\frac{1}{\sqrt{L}} \exp(i(p_k \mp m_z p'_0)x/\hbar)\}$ , respectively. Thus, both the momentum wavefunction  $|\psi_{+1}\rangle$  with the momentum eigenvalue  $(p_k - m_z p'_0)$  and  $|\psi_{-1}\rangle$  with the eigenvalue  $(p_k + m_z p'_0)$  are orthogonal to each other no matter how small the momentum value  $p'_0$  is as long as  $p'_0$  is not zero, indicating that the QUANS DAM process of (4.4) changes the initial two identical COM momentum wavefunctions ( $|\Psi_k\rangle$ ) to a pair of orthogonal wavefunctions  $\{|\psi_{+1}\rangle, |\psi_{-1}\rangle\}$ , respectively. For a finite box length  $L$ , when  $m_z p'_0 = p_l = 2\pi\hbar l/L$  ( $l \neq 0$  is a quantum number), one knows from (4.1) and (4.4) that both the wavefunctions  $\{|\psi_{+1}\rangle, |\psi_{-1}\rangle\}$  are also the momentum eigenfunctions. Then according to (4.2) they are also orthogonal to one another, indicating that the QUANS DAM process also changes the initial two identical states to a pair of orthogonal states, respectively. Here both the final states  $\{|\psi_{+1}\rangle|0\rangle, |\psi_{-1}\rangle|0\rangle\}$  have the same amplitude as the one of the initial state ( $|\Psi_k\rangle|0\rangle$ ) and keep their amplitudes unchanged during the whole QUANS DAM process. What these states are changed by the QUANS DAM process is only their spatially-dependent phases (or momentum eigenvalues). Therefore, such a QUANS DAM process is called the phase-based QUANS DAM process.

In principle, by quantum measurement one is able to distinguish the two momentum eigenfunctions  $|\psi_{+1}\rangle$  and  $|\psi_{-1}\rangle$  in (4.4b) from one another, because both the eigenfunctions are orthogonal to one another even if the momentum value  $p'_0$  takes an arbitrarily small nonzero value. Here there is no requirement that one have to measure directly the momentum eigenvalue  $p_k + m_z p'_0$  (or  $p_k - m_z p'_0$ ) so as to identify any one of the two wavefunctions  $|\psi_{+1}\rangle$  and  $|\psi_{-1}\rangle$ . Actually, both the candidate momentum eigenstates  $|\psi_{+1}\rangle|0\rangle$  and  $|\psi_{-1}\rangle|0\rangle$  could be first transferred to the two (candidate) internal energy eigenstates  $|0\rangle|\varphi_1\rangle$  and  $|0\rangle|\varphi_2\rangle$  of the single-atom system, respectively, and then by quantum measuring the atomic internal energy one is able to distinguish the two wavefunctions  $|\psi_{+1}\rangle$  and  $|\psi_{-1}\rangle$  from one another. Here  $|\varphi_1\rangle$  and  $|\varphi_2\rangle$  are the energy eigenstates of the atomic internal motion. That the two wavefunctions can be distinguished from each other unambiguously is only because the two wavefunctions are orthogonal to each other. It has nothing to do with the quantum measurement of momentum eigenvalue. Thus, rather than the quantum measurement, the phase-based QUANS DAM process really leads to that the two wavefunctions can be distinguished from each other unambiguously. Readers should be famil-

iar with the fact that when two non-orthogonal states are sufficiently close to one another, it is hard to use any quantum measurement to distinguish them from one another no matter which observable is measured.

Clearly, a phase-based QUANSDAM process is quite different from an amplitude-based QUANSDAM process and the amplitude amplification of a conventional quantum search algorithm. Like the amplitude amplification of a conventional quantum search algorithm, an amplitude-based QUANSDAM process could be affected greatly by the square speedup limit. This is a severe shortcoming for an amplitude-based QUANSDAM process. It results in that it is not easy to find an exponential amplitude-based QUANSDAM process. A phase-based QUANSDAM process seems to avoid this square-speedup-limit shortcoming. Therefore, it seems to capture the essential aspect of an exponential QUANSDAM process. Apparently its potential to realize an exponential QUANSDAM process and an exponential quantum-computing speedup seems to be infinite.

The power of a phase-based QUANSDAM process to amplify the quantum-state difference seems to be unlimited and even infinitely large in appearance. For example, the phase-based QUANSDAM process of (4.4) consisting of only one *IC* unitary propagator  $U_p^{ic}(a_m^s)$  can change the same initial states to a pair of orthogonal momentum eigenstates, respectively. This, of course, concludes merely from the principle of a phase-based QUANSDAM process in ideal conditions. In the phase-based QUANSDAM process one does not consider how many basic *IC* unitary operators are consumed to prepare the ideal *IC* unitary propagator  $U_p^{ic}(a_m^s)$  and how to prepare the ideal initial momentum eigenfunction. One does not yet take into account energy, space, and the quantum-computing resource, which could probably be used by the QUANSDAM process. For example, in practice it is impossible to prepare an ideal momentum eigenfunction which spreads over the whole coordinate space  $(-\infty, +\infty)$ . Actually the phase-based QUANSDAM process of (4.4) has not yet touched the hard problem: how to construct efficiently the phase-based QUANSDAM process itself.

The *IC* unitary propagator  $U_p^{ic}(a_m^s)$  of (4.3) may be considered as the special one of more general *IC* unitary propagators in a time- and space-dependent quantum system. The latter will be described in detail below. In practice the simple QUANSDAM processes of (3.4) and (3.12) are not useful and even the amplitude-based QUANSDAM process of (3.9) is suspected to be useful. The phase-based QUANSDAM process of (4.4) seems to be quite simple and extremely powerful, but in practice it is hard to realize. These QUANSDAM processes need to be further developed or generalized so that in practice they could be useful in a variety of quantum systems. One of the possible strategies to generalize these QUANSDAM processes is that the solution information  $(a_m^s)$  is loaded onto the time evolution propagator of a quantum system and then the generated *IC* unitary propagator acts as the basic building block to construct the desired QUANSDAM process.

One simple method to load the solution information  $(a_m^s)$  onto the unitary propagator of a quantum system with time-independent Hamiltonian  $H$  is that (1) the information-carrying unitary propagator is first written formally in the

simple form

$$U_H^{ic}(t_m, a_m^s) = \exp(-ia_m^s H t_m / \hbar) \quad (4.5)$$

where  $t_m$  is a real parameter and it is usually limited to be  $t_m \approx \theta_m$ ; (2) the *IC* unitary propagator  $U_H^{ic}(t_m, a_m^s)$  then is constructed by using the basic *IC* unitary operators and the *QM* unitary operators of the quantum system. Here a quantum system with time-independent Hamiltonian is called a time-independent quantum system. It can be found that the internal-motion-dependent *IC* unitary momentum-displacement propagator  $U_p^{ic}(a_m^s)$  of (4.3) is the special case of the *IC* unitary propagator  $U_H^{ic}(t_m, a_m^s)$ , where  $H = xS_z$  and  $t_m = p'_0$ . The basic *IC* unitary operator  $U_\lambda^{ic}(\theta_m, a_m^s)$  of (2.9) is also the special case of the *IC* unitary propagator  $U_H^{ic}(t_m, a_m^s)$ , where  $H = I_{m\lambda}$  and  $t_m/\hbar = \theta_m$ . Therefore, the *IC* unitary propagator  $U_H^{ic}(t_m, a_m^s)$  is naturally a generalization of these *IC* unitary operators  $U_p^{ic}(a_m^s)$  and  $U_\lambda^{ic}(\theta_m, a_m^s)$ . There may be a variety of quantum-mechanical methods to construct the *IC* unitary propagator of (4.5). They may include the eigenfunction expansion [17], the Green function method [14] and the Feynman path integral [18], the Trotter-Suzuki method [16], and the multiple-quantum operator algebra space [12, 22], just to name a few. But, for simplicity, below it is described only how the energy eigenfunction expansion is used to construct the *IC* unitary propagator and to realize the solution-information propagation process (See below).

Quantum mechanics provides many different methods to deal with the time evolution process of a quantum system. Two basic methods among them that have been used extensively are the eigenfunction expansion in the Hilbert space of a quantum system [17] and the Green function method [14, 18]. The eigenfunction expansion is considered as a well-established mathematical (but not a physical) principle in quantum mechanics [17, 14]. It has been used extensively to deal with the time evolution process of a quantum system and especially it is a convenient method to treat the time evolution process of a time-independent quantum system which may or may not space-dependent. The energy eigenfunction expansion is the special case of the eigenfunction expansion. Below it is used to realize the loading of the solution information ( $a_m^s$ ) onto the unitary propagator of a time-independent quantum system, that is, it is used to construct the *IC* unitary propagator of (4.5). On the other hand, the Green function method is more convenient to deal with the time evolution process of a time- and space-dependent quantum system. Both the methods have important applications in the unitary manipulation of a single atom in time and space (See Ref. [19] for the eigenfunction expansion and Refs. [31, 15, 19] for the Green function method and the Feynman path integral).

First of all, the eigenfunction expansion principle is simply introduced. For simplicity, consider a time-independent and space-dependent quantum system. Since its Hamiltonian  $H$  is time-independent, its time evolution propagator may be generally written as  $U(t) = \exp(-iHt/\hbar)$ . Suppose that  $\{u_k(\mathbf{r})\}$  is the complete orthonormal set of energy eigenfunctions of the Hilbert space of the quantum system. Then the energy-eigenvalue equation is  $Hu_k(\mathbf{r}) = E_k u_k(\mathbf{r})$ , where  $E_k$  is the energy eigenvalue associated with the energy eigenfunction

$u_k(\mathbf{r})$ . According to the eigenfunction expansion principle [17, 14] an arbitrary quantum state  $\Psi(\mathbf{r}, t_0)$  of the quantum system can be expanded in terms of the energy eigenfunctions  $\{u_k(\mathbf{r})\}$  :

$$\Psi(\mathbf{r}, t_0) = \sum_{k=0}^{\infty} A_k u_k(\mathbf{r}) \quad (4.6)$$

where  $A_k = \langle u_k | \Psi(t_0) \rangle$  is an expansion coefficient. Generally the number of expansion terms on the RH side of (4.6) is equal to the number of the energy eigenfunctions in the complete set  $\{u_k(\mathbf{r})\}$  or the dimensional size of the Hilbert space of the quantum system. It is generally infinite for a space-dependent quantum system such as a single atom motioning in one-dimensional coordinate space. However, the number of expansion terms on the RH side of (4.6) becomes finite if the number of the energy eigenfunctions is finite in the complete set  $\{u_k(\mathbf{r})\}$ . There are a lot of quantum systems for which the number of expansion terms is finite. They tend to be space-independent. One typical example is a multiple-spin-1/2 quantum system. An  $n$ -qubit quantum system also may be considered as one of these quantum systems, which has a  $2^n$ -dimensional Hilbert space. Below this type of quantum systems are treated as the special cases and will not be explicitly considered unless stated otherwise.

The expansion series on the RH side of (4.6) is always convergent no matter that its number of expansion terms is finite or infinite [17]. Now with the help of the energy-eigenvalue equation  $Hu_k(\mathbf{r}) = E_k u_k(\mathbf{r})$  it follows from (4.6) that the time evolution process of the quantum system with arbitrary initial state  $\Psi(\mathbf{r}, t_0)$  may be expressed as [17, 14]

$$\Psi(\mathbf{r}, t) = U(t) \Psi(\mathbf{r}, t_0) = \sum_{k=0}^{\infty} A_k \exp(-iE_k t/\hbar) u_k(\mathbf{r}). \quad (4.7)$$

Here without losing generality the initial time  $t_0$  is set to  $t_0 = 0$  for the time-independent quantum system. This is the basic mathematical formula for the energy eigenfunction expansion to calculate and realize the time evolution process of a time-independent quantum system. Its application to the unitary manipulation of a single atom with a Gaussian wave-packet state in time and space may be seen in Ref. [19].

Now the basic equation of (4.7) is employed to load the solution information ( $a_m^s$ ) onto the unitary propagator of a time-independent quantum system. By applying the *IC* unitary propagator  $U_H^{ic}(t_m, a_m^s)$  of (4.5) to an arbitrary state  $\Psi(\mathbf{r}, t_0)$  of the quantum system which is given by the expansion (4.6) one obtains from the basic equation of (4.7) the solution-information propagation process:

$$U_H^{ic}(t_m, a_m^s) \Psi(\mathbf{r}, t_0) = \sum_{k=0}^{\infty} A_k \exp(-ia_m^s E_k t_m/\hbar) u_k(\mathbf{r}). \quad (4.8)$$

Obviously, this solution-information propagation process is also a QUANS DAM (or UNIDYSLOCK) process. Then a unitary sequence  $USEQ(K)$  that consists

of the basic *IC* unitary operators  $\{U_{\lambda_l}^{ic}(\theta_m^l, a_m^s)\}$  and the *QM* unitary operators  $\{V_l\}$  is constructed,

$$USEQ(K) = V_K U_{\lambda_K}^{ic}(\theta_m^K, a_m^s) V_{K-1} \dots V_1 U_{\lambda_1}^{ic}(\theta_m^1, a_m^s) V_0, \quad (4.9)$$

such that when  $USEQ(K)$  is applied to the same state  $\Psi(\mathbf{r}, t_0)$ , one has

$$USEQ(K) \Psi(\mathbf{r}, t_0) = \sum_{k=0}^{\infty} A_k \exp(-i a_m^s E_k t_m / \hbar) u_k(\mathbf{r}). \quad (4.10)$$

Note that the state  $\Psi(\mathbf{r}, t_0)$  is arbitrary. By comparing (4.10) with (4.8) one finds that  $U_H^{ic}(t_m, a_m^s) = USEQ(K)$ . The equations (4.5)–(4.10) form the basis for the energy eigenfunction expansion to construct the *IC* unitary propagator of (4.5) in a time-independent quantum system which may or may not be space-dependent.

As a special case, for an  $n$ -qubit quantum system the number of expansion terms is finite on the RH side of (4.8). Then it is possible to construct a unitary sequence  $USEQ(K)$  of (4.9) with a finite number  $K$  such that  $U_H^{ic}(t_m, a_m^s) = USEQ(K)$ . For a single-atom system which is a space-dependent quantum system the number of expansion terms in (4.8) is generally infinite. Then it is generally hard to generate a unitary sequence of (4.9) such that  $U_H^{ic}(t_m, a_m^s) = USEQ(K)$  exactly, but it is still possible to construct a unitary sequence  $USEQ(K)$  with a finite number  $K$  such that  $USEQ(K)$  is equal to  $U_H^{ic}(t_m, a_m^s)$  approximately. Here the discussion is not involved in the computational complexity of the unitary sequence  $USEQ(K)$ . However, if the *IC* unitary propagator  $U_H^{ic}(t_m, a_m^s)$  is used as the basic building block of a QUANSDAM process, then one must consider the computational complexity of the unitary sequence  $USEQ(K)$  that generates  $U_H^{ic}(t_m, a_m^s)$  exactly or approximately. From the point of view of the computational complexity it is required that the *IC* unitary propagator  $U_H^{ic}(t_m, a_m^s)$  be generated by the unitary sequence  $USEQ(K)$  of (4.9) with a minimum number  $K$  of the basic *IC* unitary operators and moreover, the number  $K$  be polynomially large at most.

Because the RH side of (4.8) is an infinite series, generally the *IC* unitary propagator  $U_H^{ic}(t_m, a_m^s)$  could only be constructed approximately for a space-dependent quantum system. Note that the infinite series of (4.8) also is a QUANSDAM process. As far as the QUANSDAM process is concerned, it is more convenient and direct to realize approximately the solution-information propagation process  $U_H^{ic}(t_m, a_m^s) \Psi(\mathbf{r}, t_0)$  of (4.8) than to construct approximately the *IC* unitary propagator  $U_H^{ic}(t_m, a_m^s)$  separately. According to the eigenfunction expansion principle [17] the infinite series on the RH side of (4.6) is always convergent. This is the theoretical basis for employing the energy eigenfunction expansion to construct approximately the *IC* unitary propagator  $U_H^{ic}(t_m, a_m^s)$  and to realize approximately the solution-information propagation process of (4.8). The eigenfunction expansion principle itself is not involved in the convergent speed of the infinite series of (4.6). But the latter is important to determine whether the *IC* unitary propagator  $U_H^{ic}(t_m, a_m^s)$  can be efficiently



constructed approximately and whether the solution-information propagation process of (4.8) can be efficiently realized approximately.

For simplicity, below consider a single-atom system with an external harmonic potential field. Then the convergent speed of the infinite series of (4.6) may be simply characterized through the truncation error:

$$\varepsilon(M) = \left\| \sum_{k=0}^{M-1} A_k u_k(\mathbf{r}) - \Psi(\mathbf{r}, t_0) \right\| = \sqrt{\sum_{k=M}^{\infty} |A_k|^2}. \quad (4.11a)$$

Given any desired small value  $\varepsilon > 0$  the convergence of the infinite series of (4.6) is said to be faster if the minimum number  $M$  of the expansion terms that satisfies  $\varepsilon(M) < \varepsilon$  is smaller. Obviously, the convergent speed is directly dependent on the expansion coefficients  $\{A_k\}$ . It can prove that if  $M$  is polynomially large and the truncation error  $\varepsilon(M) < \varepsilon$  can be neglected, then there is an efficient unitary sequence  $USEQ(K)$  of (4.9) such that the solution-information propagation process of (4.8) can be realized by the one of (4.10) up to a global phase factor. The truncation error  $\varepsilon(M)$  of (4.11a) may act as a simple fast-convergent criterion for the infinite series of (4.6). It may be better used for the special case that contribution of the expansion terms with the lowest energy eigenstates to the series of (4.6) is dominating. Though the fast-convergent criterion of (4.11a) is simple and special, it could be used not only for the single-atom system but also for other space-dependent quantum systems. For a more general case the double-side truncation error may act as a better fast-convergent criterion for the infinite series of (4.6). It is defined by

$$\varepsilon(L, M) = \left\| \sum_{k=L}^{L+M-1} A_k u_k(\mathbf{r}) - \Psi(\mathbf{r}, t_0) \right\| = \sqrt{\sum_{k=0}^{L-1} |A_k|^2 + \sum_{k=L+M}^{\infty} |A_k|^2}. \quad (4.11b)$$

When the energy quantum number  $L = 0$ , this formula is reduced to (4.11a). Thus, the fast-convergent criterion of (4.11a) is the special case of the general one of (4.11b). It can turn out that if  $M$  is polynomially large and for any given quantum number  $L \geq 0$  the double-side truncation error  $\varepsilon(L, M) < \varepsilon$  can be neglected, then there exists an efficient unitary sequence  $USEQ(K)$  of (4.9) such that the solution-information propagation process of (4.8) can be realized by the one of (4.10) up to a global phase factor.

If any quantum state  $\Psi(\mathbf{r}, t_0)$  whose energy eigenfunction expansion is given by the infinite series of (4.6) satisfies the fast-convergent criterion  $\varepsilon(L, M) < \varepsilon$ , where the quantum number  $L \geq 0$  is some integer and  $M$  is polynomially large, then the state is called a fast-convergent quantum state. Because the quantum number  $L \geq 0$  in (4.11b) may be arbitrarily large and the polynomially large number  $M$  is not fixed, there are an infinite number of linearly independent fast-convergent quantum states of the single-atom system. All these fast-convergent quantum states form a fast-convergent state set. Evidently the fast-convergent state set is a state subset of the Hilbert space of the single-atom system. It

contains an infinite number of linearly independent fast-convergent quantum states.

One important point is that the fast-convergent criterion  $\varepsilon(L, M) < \varepsilon$  has nothing to do with any quantum-state effect of the single-atom quantum system, because a wavefunction  $\Psi(\mathbf{r}, t_0)$  of the quantum system which is represented by the expansion of (4.6) and satisfies the fast-convergent criterion may be quantum or classical. Therefore, whether or not the wavefunction  $\Psi(\mathbf{r}, t_0)$  is a fast-convergent state has nothing to do with any quantum-state effect of the quantum system. Moreover, it has nothing to do with the energy, space, and even the fundamental quantum-computing resource, which are associated with the unitary sequence  $USEQ(K)$  of (4.9). It also has nothing to do with whether the solution-information propagation process of (4.8) is a QUANSDAM process or a UNIDYSLOCK process, when the wavefunction contains the solution information  $(a_m^s)$ . Actually, it is determined completely by the shape of the wavefunction for a given complete set of energy eigenfunctions  $\{u_k(\mathbf{r})\}$ .

The *IC* unitary propagator  $U_H^{ic}(t_m, a_m^s)$  may act as the basic building block to construct a QUANSDAM process. By replacing the *IC* unitary operator  $V_k^{ic}(a_m^s)$  with  $U_H^{ic}(t_m^k, a_m^s)$  in the general QUANSDAM process of (3.8) one obtains

$$QUANSDAM(K, a_m^s) = U_K U_H^{ic}(t_m^K, a_m^s) U_{K-1} \dots U_1 U_H^{ic}(t_m^1, a_m^s) U_0 \quad (4.12)$$

where the initial state which is omitted may be simply set to the ground state of the quantum system and  $U_k$  may be the *QM* unitary operator or propagator of the quantum system.

The total *QSD* amplification ability of the QUANSDAM process of (4.12) is determined by (i) the unitary sequence  $USEQ(K)$  of (4.9) that is used to realize the *IC* unitary propagator  $U_H^{ic}(t_m^k, a_m^s)$  and (ii) the *QSD* amplification ability of each *IC* unitary propagator  $U_H^{ic}(t_m^k, a_m^s)$  that is promoted by the *QM* unitary propagators  $\{U_k\}$ .

Because the parameter  $t_m^k$  ( $t_m^k \approx \theta_m$ ) in the *IC* unitary propagator  $U_H^{ic}(t_m^k, a_m^s)$  is exponentially small, the *QSD* amplification ability of the *IC* unitary propagator alone is weak and generally an *IC* unitary propagator  $U_H^{ic}(t_m^k, a_m^s)$  alone is not able to realize an exponential *QSD* amplification. Then the interaction between the *IC* unitary propagators  $\{U_H^{ic}(t_m^k, a_m^s)\}$  and the *QM* unitary propagators  $\{U_k\}$  may be more important to realize an exponential *QSD* amplification of the QUANSDAM process of (4.12). This interaction could be stronger than the one between the basic *IC* unitary operators and the *QM* unitary operators in a QUANSDAM process such as (3.9). However, from the point of view of the computational complexity the realization of an exponential *QSD* amplification makes sense only when the *IC* unitary propagator  $U_H^{ic}(t_m^k, a_m^s)$  can be generated by the efficient unitary sequence  $USEQ(K)$  of (4.9) or the solution-information propagation process of (4.8) can be realized efficiently.

As pointed out in the preceding section, an appropriate *QM* unitary operator in (3.9) is able to make the *QSD* varying rate  $\Delta\rho_{12}(k+1)/k$  (or  $\Delta\rho_{12}^2(k+1)/k$ ) in magnitude exceeding  $O(p_2(k))$ , where  $p_2(k)$  is a quadratic polynomial in the

number  $k$  of the basic  $IC$  unitary operators in (3.9). But for the QUANS DAM process of (4.12) the basic building block is directly the  $IC$  unitary propagator  $U_H^{ic}(t_m^k, a_m^s)$  rather than the basic  $IC$  unitary operator. For simplicity, here suppose that the  $IC$  unitary propagator  $U_H^{ic}(t_m^k, a_m^s)$  or the solution-information propagation process of (4.8) can be realized efficiently. Then a  $QM$  unitary propagator (e.g.,  $U_k$ ) in (4.12) is said to be appropriate if it can make the  $QSD$  varying rate  $\Delta\rho_{12}(k+1)/k$  (or  $\Delta\rho_{12}^2(k+1)/k$ ) exceeding  $O(p_2(k))$  in magnitude, where  $p_2(k)$  is a quadratic polynomial in the number  $k$  of the  $IC$  unitary propagators  $\{U_H^{ic}(t_m^k, a_m^s)\}$  in the process  $QUANS DAM(k, a_m^s)$  ( $0 \leq k < K$ ) of (4.12). If every  $QM$  unitary propagator in (4.12) is appropriate, then clearly the QUANS DAM process of (4.12) can be used to extract the solution information in a super-square or even an exponential speedup. Then these quantum states are clearly appropriate, which are  $|\Psi_{a_m^s}^k\rangle = QUANS DAM(k, a_m^s)|\Psi_0\rangle$  ( $0 \leq k \leq K$ ) with the initial state  $|\Psi_0\rangle$  which usually may be the ground state of the quantum system, if every  $QM$  unitary propagator in (4.12) is appropriate. It is obvious that each one of these appropriate states  $\{|\Psi_{a_m^s}^k\rangle\}$  is a fast-convergent state.

How to find the appropriate  $QM$  unitary propagators and the appropriate states of the QUANS DAM process of (4.12) is a challenging task. These appropriate  $QM$  unitary propagators must ensure that on the one hand, the  $QSD$  amplification ability of the QUANS DAM process is super-square or even exponentially large and on the other hand, the solution-information propagation process of (4.8) can be realized efficiently. Here the unitary manipulation of a single atom in time and space [15, 31, 19, 7] that is based on Gaussian wave-packet states and quadratic unitary propagators is a systematic and powerful method to find these appropriate  $QM$  unitary propagators and these appropriate states for the QUANS DAM process in a single-atom system. The related work will be reported in detail in future.

## 5. The $HSSS$ quantum search process

Consider an unstructured search problem whose unstructured search space is  $2^n$ -dimensional and which may be solved in an  $n$ -qubit quantum system. This search problem may be fully characterized by the Boolean function  $f : \{0, 1, \dots, 2^n - 1\} \rightarrow \{0, 1\}$  [13]. Now the unstructured search space of the search problem may be given by  $\{x\} = \{0, 1, \dots, 2^n - 1\}$  (e.g.,  $x$  may be an index number). The unique solution  $x_0$  to the search problem,  $x_0 \in \{0, 1, \dots, 2^n - 1\}$ , may be defined simply by the Boolean function  $f(x) = 1$  if  $x = x_0$  and  $f(x) = 0$  if  $x \neq x_0$ . It also may be defined equivalently by the Boolean functional operation:

$$U_f : |x\rangle|0\rangle \rightarrow |x\rangle|f(x)\rangle = \begin{cases} |x_0\rangle|1\rangle & \text{if } x = x_0 \\ |x\rangle|0\rangle & \text{if } x \neq x_0 \end{cases} \quad (5.1)$$

It is clear that the solution space (or set) to the search problem is  $\{|x_0\rangle|f(x_0)\rangle\}$  or  $\{|x_0\rangle|1\rangle\}$  with  $0 \leq x_0 \leq 2^n - 1$ , while the unstructured search space is  $\{|x\rangle\}$  with  $0 \leq x \leq 2^n - 1$ . If one drops the functional state  $|f(x_0)\rangle = |1\rangle$  in the second register, then the solution space is just  $\{|x_0\rangle\}$  with  $0 \leq x_0 \leq 2^n - 1$ . Here the

solution  $|x_0\rangle$  can be any element of the solution space  $\{|x_0\rangle\}$ . Thus, the solution space is just the unstructured search space. Without confusion, here  $|x\rangle$  denotes the variable (number) state in the register and also the usual computational basis vector in the math Hilbert space corresponding to the register, as can be seen below. A similar denotation is also available for  $|f(x)\rangle$ .

According to the Lecerf-Bennett reversible computational theory the Boolean functional operation  $U_f$  of (5.1) may be constructed in a reversible form [8, 27]. Once it is constructed in this way, it may be applied to a quantum system. This construction usually needs to use the intermediate reversible computational steps and the auxiliary registers which are already hidden in (5.1). Now the Boolean functional operation  $U_f$  may be performed in an  $n$ -qubit quantum system with the variable state space  $\{|x\rangle\}$  and a functional quantum system with the functional state space  $\{|f(x)\rangle\}$  together. Here every intermediate reversible computational step may be replaced with the corresponding unitary operator (or quantum gate array [35]) that acts on the total quantum system which may include the auxiliary quantum systems in addition to the  $n$ -qubit quantum system and the functional quantum system; and in (5.1) the variable state  $|x\rangle$  and the functional state  $|f(x)\rangle$  in the registers are replaced with the usual computational basis state  $|x\rangle$  of the  $n$ -qubit quantum system and the functional state  $|f(x)\rangle$  of the functional quantum system, respectively.

Here one needs to pay attention to the difference between the solution space (or set)  $\{|x_0\rangle\}$ , the unstructured search space  $\{|x\rangle\}$ , and the usual Hilbert space  $\{|x\rangle\}$  of the quantum system. In classical search the unstructured search space  $\{|x\rangle\}$  is usually just the solution set  $\{|x_0\rangle\}$ . In a usual quantum search algorithm it may be considered as the Hilbert space  $\{|x\rangle\}$  of the  $n$ -qubit quantum system [3, 13] and the solution set is the usual computational basis state set of the Hilbert space. Note that the Hilbert space is determined completely by a basis state set. Then the unstructured search space, i.e., the Hilbert space is still determined completely by the Boolean function  $f(x)$  above [13]. In the *HSSS* quantum search process the situation may be different. The unstructured search space may be a linear complex vector space  $\{|x\rangle\}$  (i.e., a math Hilbert space) and the solution set is the usual computational basis vector set of the vector space. This linear complex vector space has a similar mathematical definition as the usual Hilbert space of a quantum system and it is unstructured, but it is mathematical and does not have any physical meaning. Again the unstructured search space, i.e., the math Hilbert space is still determined completely by the Boolean function  $f(x)$  above, because a linear vector space is determined completely by a basis vector set. Therefore, there are different considerations of the unstructured search space for the classical search algorithm, the usual quantum search algorithm, and the *HSSS* quantum search process, respectively.

The mathematical Hilbert space (i.e., the math Hilbert space) is a fundamental concept in the quantum-computing speedup theory [1]. It is already introduced simply in the section 2 above. The concept of the math Hilbert space does not exist independently in conventional quantum computation. The reversible Boolean functional operation  $U_f$  of (5.1) also may be performed in the total math Hilbert space. It is known from (5.1) that the Boolean functional

operation works in the pivotal register with the variable (number) state space  $\{|x\rangle\}$ , the pivotal register with the functional (number) state space  $\{|f(x)\rangle\}$ , and the auxiliary registers together. Now every register is replaced with a linear complex vector space, i.e., a math Hilbert space. Then the Boolean functional operation may be performed in the total math Hilbert space which is the tensor product of the component math Hilbert spaces that correspond to the register with  $\{|x\rangle\}$ , the register with  $\{|f(x)\rangle\}$ , and the auxiliary registers, respectively. The component math Hilbert spaces each are large enough so that they can accommodate the variable vector space  $\{|x\rangle\}$ , the functional vector space  $\{|f(x)\rangle\}$  and so on, respectively.

Generally, in the quantum-computing speedup theory a math Hilbert space is not equivalent to the corresponding Hilbert space of a quantum system. A mathematical-parallel computation (or operation) is allowed to perform in a math Hilbert space. In effect a mathematical-parallel operation is just a usual classical-parallel operation. It is essentially different from a usual quantum-parallel operation [11] as it may not obey the quantum superposition principle.

Now one may perform the Boolean functional operation of (5.1) in the math Hilbert space of the search problem or in the Hilbert space of the quantum system. If the two Hilbert spaces are treated as the same one (See below more clearly), then this is the routine treatment of a conventional quantum search algorithm [3, 13]. If the two Hilbert spaces are treated separately, then this is the treatment of the *HSSS* quantum search process.

It is more general to use directly an oracle operation rather than a usual functional operation as the basic building block of a quantum search algorithm [3, 13]. Generally, in quantum computation an oracle operation also may be thought of as a black-box functional operation. Moreover, it could be constructed with the usual functional operations of a computational problem to be solved. Below a black-box functional operation is constructed by using the Boolean functional operation of (5.1). It may be further used as the basic building block to construct a quantum search algorithm to solve the unstructured search problem.

By using the Boolean functional operation  $U_f$  of (5.1) one carries out a sequence of the Boolean functional operations [5a, 1, 2]:

$$\begin{aligned}
V_0 &: |x\rangle|0\rangle|0\rangle \rightarrow |x\rangle|0\rangle|1\rangle \\
U_f &: |x\rangle|0\rangle|1\rangle \rightarrow |x\rangle|f(x)\rangle|1\rangle \\
V(\theta) &: |x\rangle|f(x)\rangle|1\rangle \rightarrow \exp(-i\theta\delta(f(x), 1))|x\rangle|f(x)\rangle|1\rangle \\
U_f &: \exp(-i\theta\delta(f(x), 1))|x\rangle|f(x)\rangle|1\rangle \rightarrow \exp(-i\theta\delta(f(x), 1))|x\rangle|0\rangle|1\rangle \\
V_0 &: \exp(-i\theta\delta(f(x), 1))|x\rangle|0\rangle|1\rangle \rightarrow \exp(-i\theta\delta(f(x), 1))|x\rangle|0\rangle|0\rangle
\end{aligned}$$

where the  $\delta$ -function  $\delta(f(x), 1) = 1$  if the Boolean function  $f(x) = 1$  and  $\delta(f(x), 1) = 0$  if  $f(x) = 0$ . The Boolean function-operational sequence above may be written in a compact form

$$BFSEQ = V_0 U_f V(\theta) U_f V_0 \quad (5.2)$$

where  $V_0 = \exp(-i\pi/2)\exp(i\pi I_x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $V(\theta)$  is the unitary diagonal matrix  $V(\theta) = \text{Diag}(1, 1, 1, \exp(-i\theta))$ . The Boolean function-operational sequence *BFSEQ* is executed in the main quantum system ( $|x\rangle$ ) and the two auxiliary qubits with the initial state set to  $|0\rangle|0\rangle$ . Here the single-qubit unitary operation  $V_0$  is performed only in the second auxiliary qubit and the two-qubit phase-shift operation  $V(\theta)$  is performed only in the two auxiliary qubits. If now all the intermediate steps are hidden, then the above function-operational process to perform the sequence *BFSEQ* may be simply written as

$$BFSEQ : |x\rangle|0\rangle|0\rangle \rightarrow \begin{cases} \exp(-i\theta)|x\rangle|0\rangle|0\rangle & \text{if } f(x) = 1 \\ |x\rangle|0\rangle|0\rangle & \text{if } f(x) = 0 \end{cases} \quad (5.3)$$

The core of the Boolean function-operational sequence *BFSEQ* is the Boolean functional operation  $U_f$ . The sequence *BFSEQ* may act as the black-box functional operation of a black-box quantum search algorithm to solve an unstructured search problem. This black-box functional operation is defined by the unitary transformation of (5.3) and is realized by the Boolean function-operational sequence of (5.2). Now it is very clear that both the unstructured search space  $\{|x\rangle\}$  and the Hilbert space  $\{|x\rangle\}$  of the  $n$ -qubit quantum system are the same one! This black-box functional operation is characterized completely by the mathematical-logical principle of the unstructured search problem or equivalently by the Boolean function  $f : \{0, 1, \dots, 2^n - 1\} \rightarrow \{0, 1\}$  for which there is a unique element  $x_0$  such that  $f(x_0) = 1$  [13]. In the unitary transformation of (5.3) the solution state  $|x\rangle = |x_0\rangle$  of the search problem which satisfies  $f(x) = 1$  if  $x = x_0$  and  $f(x) = 0$  if  $x \neq x_0$  can be an arbitrary computational basis state of the unstructured search space  $\{|x\rangle\}$ .

The black-box functional operation of (5.3) may be further used as the basic building block to construct a quantum search algorithm to solve the unstructured search problem. If now one drops the auxiliary qubits ( $|0\rangle|0\rangle$ ) from (5.3), then the black-box functional operation is just the usual oracle operation  $U_o(\theta)$  defined by (2.2) of a conventional quantum search algorithm. Therefore, the usual oracle operation  $U_o(\theta)$  of (2.2) can represent faithfully the black-box functional operation of (5.3).

It is known that the Boolean function  $f(x)$  in (5.3) is defined by  $f(x) = 1$  if and only if  $x = x_0$  and  $f(x) = 0$  if  $x \neq x_0$ . If one hides the auxiliary qubits ( $|0\rangle|0\rangle$ ), then the black-box functional operation of (5.3) also may be expressed as

$$BFSEQ : |x\rangle \rightarrow \begin{cases} \exp(-i\theta)|x\rangle & \text{if } x = x_0 \\ |x\rangle & \text{if } x \neq x_0 \end{cases} \quad (5.4)$$

Later it will be shown that the black-box functional operation of (5.4) also may be represented faithfully by the unitary oracle selective diagonal operator  $C_S(\theta)$  defined by (2.1) of the *HSSS* quantum search process.

As pointed out in the quantum-computing speedup theory [1], the *HSSS* quantum search process is essentially different from a conventional quantum search algorithm [3, 13]. Below it is shown from the mathematical-logical viewpoint that the unitary oracle selective diagonal operator  $C_S(\theta)$  of the *HSSS*

quantum search process is essentially different from the usual oracle operation  $U_o(\theta)$  of a conventional quantum search algorithm. Here, as is well known, the unitary oracle selective diagonal operator  $C_S(\theta)$  and the usual oracle operation  $U_o(\theta)$  characterize essentially the *HSSS* quantum search process and the conventional quantum search algorithm, respectively.

First of all, it must be pointed out that a unitary selective diagonal operator  $C_k(\theta) = \exp(-i\theta D_k)$  itself (See Refs. [5, 25b, 12] and also the section 2 above) does not own a specific mathematical-logical meaning. From the mathematical-logical viewpoint it has nothing to do with the oracle operation  $U_o(\theta)$  of a conventional quantum search algorithm, because the latter has an unambiguous mathematical-logical meaning. Its mathematical-logical meaning comes from the computational problem to be solved, when it is used to solve the problem. It could be used not only for solving an unstructured search problem [5] but also for solving other hard problems (See, for example, Ref. [25b]). If now the unitary selective diagonal operator  $C_k(\theta)$  is employed to solve an unstructured search problem, then it is given the mathematical-logical meaning of the search problem, and here it represents the black-box functional operation of the search problem. Suppose that  $C_k(\theta)$  represents the black-box functional operation of a conventional quantum search algorithm [3, 13], then it is just the usual oracle operation  $U_o(\theta)$ . If here  $C_k(\theta)$  represents the black-box functional operation of the *HSSS* quantum search process, then it is renamed the unitary oracle selective diagonal operator  $C_S(\theta)$ . In this paper  $C_S(\theta)$  represents only the black-box functional operation of the *HSSS* quantum search process. Initially the author often explained misunderstandingly the diagonal matrix of  $C_S(\theta)$  as the matrix representation of the usual oracle operation  $U_o(\theta)$ . However, it has been firmly recognized by the author since its birth's day [5a] that the *HSSS* quantum search process is essentially different from a conventional quantum search algorithm. Therefore, this misunderstanding explanation has never obstructed the author's insistent effort to realize the search-space dynamical reduction in the past one and half decades [5, 6, 1, 2]. The search scheme to solve an unstructured search problem for a conventional quantum search algorithm is to find directly the solution state  $|S\rangle$  as a whole. In contrast, the search scheme for the *HSSS* quantum search process is first to determine directly the information  $(a_k^s)$  of each component state  $(|s_k\rangle)$  of the solution state  $|S\rangle = |s_1\rangle|s_2\rangle\cdots|s_n\rangle$ , then obtain the unit-number vector  $\{a_1^s, a_2^s, \dots, a_n^s\}$ , and finally find the solution state  $|S\rangle$  of the unstructured search problem. Therefore, what the *HSSS* quantum search process treats is any single component state of the solution state, while what the conventional quantum search algorithm treats is the solution state as a whole.

In classical computation a parallel computation always can be expressed as a consecutive series of computations. This principle is also correct in quantum computation, that is, a quantum parallel functional operation [11, 10, 9] also can be expressed as a consecutive series of single (or selective) functional operations. Here this principle is simply introduced so as to help ones understand better the mathematical-logical difference between the unitary oracle selective diagonal operator  $C_S(\theta)$  and the usual oracle operation  $U_o(\theta)$ . It works in the frame

of unitary quantum dynamics. It has nothing to do with any quantum-state effects such as the quantum-state superposition, coherence interference, entanglement and nonlocal effect, and correlation which are the core components in the quantum parallel principle [11a]. The quantum-computing speedup theory [1] considers that a quantum parallel operation belongs to the unitary quantum dynamics that could be a driving force to speed up a quantum computation, but it doesn't think that these core components of the quantum parallel principle are responsible for an essential quantum-computing speedup.

The quantum parallel operation  $V_f$  of a function  $f(x)$  in an  $n$ -qubit quantum system may be generally expressed as [11a]

$$V_f : \sum_{x \in \{0,1\}^n} a_x |x\rangle |y\rangle \rightarrow \sum_{x \in \{0,1\}^n} a_x |x\rangle |y \oplus f(x)\rangle \quad (5.5)$$

where the usual computational basis state  $|x\rangle = |b_1^x, b_2^x, \dots, b_n^x\rangle$  with  $b_k^x \in \{0, 1\}$  ( $1 \leq k \leq n$ ). The initial state on the left-hand side of (5.5) is an arbitrary state of the quantum system. It is expanded in terms of the basis states  $\{|x\rangle |y\rangle\}$  of the Hilbert space (or its subspace) of the quantum system according to the eigenfunction expansion principle [17, 14]. A selective (or single) reversible functional operation of the function  $f(x)$  in the quantum system may be defined as

$$V_{f(x_1)} : \sum_{x \in \{0,1\}^n} a_x |x\rangle |y\rangle \rightarrow \sum_{x \in \{0,1\}^n, x \neq x_1} a_x |x\rangle |y\rangle + a_{x_1} |x_1\rangle |y \oplus f(x_1)\rangle. \quad (5.6)$$

This single reversible functional operation  $V_{f(x_1)}$  really performs selectively the reversible functional operation:

$$V_{f(x_1)} : |x_1\rangle |y\rangle \rightarrow |x_1\rangle |y \oplus f(x_1)\rangle \quad (5.7)$$

with the selected variable value  $x_1$ . For any other computational basis state  $|x\rangle \neq |x_1\rangle$  of the quantum system (or for any other variable value  $x \neq x_1$ ) the single reversible functional operation makes no action. The selective functional operation  $V_{f(x_1)}$  is considered as a single reversible functional operation rather than a parallel functional operation, although the initial state of the functional operation may be a superposition state, because one action of the functional operation on the initial state computes only one functional value, i.e.,  $x_1 \rightarrow f(x_1)$ .

The construction of a selective reversible functional operation may be simple or complex in a quantum system, but this doesn't matter for the present purpose. For simplicity, here assume that any two selective reversible functional operations do not have a cross-talk. For example, one has

$$\begin{aligned} V_{f(x_2)} V_{f(x_1)} : \sum_{x \in \{0,1\}^n} a_x |x\rangle |y\rangle &\rightarrow \sum_{x \in \{0,1\}^n, x \neq x_1, x_2} a_x |x\rangle |y\rangle \\ &+ a_{x_2} |x_2\rangle |y \oplus f(x_2)\rangle + a_{x_1} |x_1\rangle |y \oplus f(x_1)\rangle. \end{aligned} \quad (5.8)$$



This formula shows that the second selective reversible functional operation  $V_{f(x_2)}$  does not affect the function-operational result  $|x_1\rangle|y \oplus f(x_1)\rangle$  of the first selective reversible functional operation  $V_{f(x_1)}$ . Now using this no-cross-talk assumption and noting that the initial state  $\sum_{x \in \{0,1\}^n} a_x |x\rangle|y\rangle$  is arbitrary it can turn out that the relation between the quantum parallel functional operation  $V_f$  of (5.5) and the single reversible functional operations  $\{V_{f(x)}\}$  of (5.6) may be simply written as

$$V_f = \prod_{x \in \{0,1\}^n} V_{f(x)}. \quad (5.9)$$

This formula shows that the quantum parallel functional operation  $V_f$  may be decomposed as a consecutive sequence of many ( $2^n$ ) single reversible functional operations  $\{V_{f(x)}\}$ . For this result a quantum parallel functional operation is similar to the classical counterpart. The formula (5.9) is independent of any initial state in (5.5) and it is purely unitary-dynamical. The superposition of functional states on the right-hand side of (5.5) may be generated by applying the quantum parallel operation  $V_f$  to the initial state in (5.5) or by applying the consecutive sequence of the single reversible functional operations on the right-hand side of (5.9) to the same initial state. Obviously, the quantum parallel functional operation  $V_f$  is greatly economic in the generation of the superposition of functional states in (5.5) when there are exponentially many single reversible functional operations  $\{V_{f(x)}\}$  on the right-hand side of (5.9).

The relation (5.9) between the quantum parallel functional operation  $V_f$  and the selective reversible functional operations  $\{V_{f(x)}\}$  is helpful for understanding correctly the mathematical-logical difference between the unitary oracle selective diagonal operator  $C_S(\theta)$  and the usual oracle operation  $U_o(\theta)$ . Suppose that the quantum parallel functional operation  $V_f$  as a whole is irreducible. Then whether or not any selective reversible functional operation  $V_{f(x)}$  in (5.9) must be irreducible too? Actually, the relation (5.9) does not require that each of the selective reversible functional operations  $\{V_{f(x)}\}$  be irreducible. Thus, any single reversible functional operation  $V_{f(x)}$  may or may not be irreducible, even if the quantum parallel functional operation  $V_f$  as a whole is irreducible. This property is very helpful for understanding correctly why the unitary oracle selective diagonal operator  $C_S(\theta)$  is reducible, while the usual oracle operation  $U_o(\theta)$  is not. Later this point will be discussed. It must be emphasized that here the so-called ‘irreducible’ (or ‘reducible’) must be related to the mathematical-logical meaning of a functional operation (or a computational problem).

With the help of the quantum parallel functional operation and the selective reversible functional operation, below from the mathematical-logical viewpoint it is shown that the unitary oracle selective diagonal operator  $C_S(\theta)$  is essentially different from the usual oracle operation  $U_o(\theta)$ . There is no doubt that when a quantum algorithm solves a computational problem, it must obey the mathematical-principle of the computational problem. However, it is allowed in mathematics and computation that a same computational problem could be solved by different algorithms. The *HSSS* quantum search process and a conventional quantum search algorithm are just two different quantum search algorithms to solve a same unstructured search problem. If they solve

the same search problem, then they must obey the same mathematical-logical principle of the search problem. For a conventional quantum search algorithm the mathematical-logical principle of an unstructured search problem is characterized essentially by the usual oracle operation  $U_o(\theta)$  defined by (2.2). It is shown above that the usual oracle operation  $U_o(\theta)$  is equivalent to the black-box functional operation of (5.3) or (5.4) that is generated by the Boolean functional operation  $U_f$  of (5.1). For the *HSSS* quantum search process the mathematical-logical principle is characterized essentially by the unitary oracle selective diagonal operator  $C_S(\theta)$  defined by (2.1). Below it must prove that the unitary oracle selective diagonal operator  $C_S(\theta)$  is also equivalent to the black-box functional operation of (5.3) or (5.4).

It is known that the usual oracle operation  $U_o(\theta)$  has the mathematical-logical meaning of quantum parallel operation [3, 13]. In contrast, according to the expression  $C_S(\theta) = \exp(-i\theta D_S)$  ( $D_S = |S\rangle\langle S|$  and  $|S\rangle$  is the solution state) in (2.1) the unitary oracle selective diagonal operator  $C_S(\theta)$  may not have. Moreover, the usual oracle operation  $U_o(\theta)$  as a whole is irreducible, while  $C_S(\theta)$  is reducible. How can they each characterize essentially the same mathematical-logical principle of the search problem?

Because the black-box functional operation of (5.4) is a quantum parallel operation which is applied to an  $n$ -qubit quantum system, just like the quantum parallel functional operation  $V_f$  of (5.9) it may be decomposed as a consecutive sequence of many selective reversible functional operations. According to (5.4) one may construct any selective black-box functional operation  $BFSEQ(y, \theta)$  by

$$BFSEQ(y, \theta) : \sum_{x \in \{0,1\}^n} b_x |x\rangle \rightarrow \sum_{x \in \{0,1\}^n, x \neq x_0} b_x |x\rangle + b_{x_0} \exp(-i\theta \delta(y - x_0)) |x_0\rangle \quad (5.10)$$

where the  $\delta$ -function  $\delta(y - x_0) = 1$  if  $y = x_0$  and  $\delta(y - x_0) = 0$  if  $y \neq x_0$ , and  $|x_0\rangle$  is the solution state of the search problem. The initial state on the left-hand side of (5.10) is arbitrary in the quantum system. It is expanded in terms of the basis states  $\{|x\rangle\}$  of the Hilbert space of the quantum system according to the eigenfunction expansion principle. Here it is required that any two selective black-box functional operations of (5.10) do not have a cross-talk. A particularly important selective black-box functional operation of (5.10) is  $BFSEQ(x_0, \theta)$  which is selectively applied to the solution state  $|x_0\rangle$ . It is obtained from (5.10) by setting  $y = x_0$ ,

$$BFSEQ(x_0, \theta) : \sum_{x \in \{0,1\}^n} b_x |x\rangle \rightarrow \sum_{x \in \{0,1\}^n, x \neq x_0} b_x |x\rangle + b_{x_0} \exp(-i\theta) |x_0\rangle. \quad (5.11)$$

This selective functional operation alone performs the reversible functional operation in the quantum system:

$$BFSEQ(x_0, \theta) : |x_0\rangle \rightarrow \exp(-i\theta) |x_0\rangle \quad (5.12)$$

with the solution state  $|x_0\rangle$ . Besides  $BFSEQ(x_0, \theta)$  any other selective black-box functional operation with  $y \neq x_0$  also can be obtained from (5.10) and it is

given by

$$BFSEQ(y, \theta) : \sum_{x \in \{0,1\}^n} b_x |x\rangle \rightarrow \sum_{x \in \{0,1\}^n} b_x |x\rangle, \text{ for } y \neq x_0. \quad (5.13)$$

This selective reversible functional operation as a whole is a unit operator. But its content could not be always identical to a unit operator. On the basis of the no-cross-talk assumption it turns out that the quantum parallel black-box functional operation of (5.4) may be written as a consecutive sequence of the single black-box functional operations  $\{BFSEQ(y, \theta)\}$  defined by (5.10):

$$BFSEQ = \prod_{y \in \{0,1\}^n} BFSEQ(y, \theta). \quad (5.14)$$

This formula is similar to (5.9). It is independent upon any quantum state of the quantum system and hence it is purely unitary-dynamical. Actually, with the help of (5.11) and (5.13) one can prove that the consecutive sequence of (5.14) applying to an arbitrary initial state  $\sum_{x \in \{0,1\}^n} b_x |x\rangle$  of the quantum system generates just the same state as the black-box functional operation of (5.4) applying to the same initial state does,

$$BFSEQ : \sum_{x \in \{0,1\}^n} b_x |x\rangle \rightarrow \sum_{x \in \{0,1\}^n, x \neq x_0} b_x |x\rangle + b_{x_0} \exp(-i\theta) |x_0\rangle. \quad (5.15)$$

It is easy to find from (5.11) and (5.15) that both the quantum parallel black-box functional operation ( $BFSEQ$ ) of (5.4) and the selective black-box functional operation  $BFSEQ(x_0, \theta)$  that is selectively applied to the solution state  $|x_0\rangle$  generate the same unitary transformation of (5.4). This is the very reason why in the past the author explained misunderstandingly the diagonal matrix of  $C_S(\theta)$  as the matrix representation of the usual oracle operation  $U_o(\theta)$ . This point will be seen more clearly below.

According to (2.2) the usual oracle operation  $U_o(\theta)$  applying to an arbitrary initial state of the quantum system generates

$$U_o(\theta) : \sum_{x \in \{0,1\}^n} b_x |x\rangle \rightarrow \sum_{x \in \{0,1\}^n, x \neq x_0} b_x |x\rangle + b_{x_0} \exp(-i\theta) |x_0\rangle. \quad (5.16)$$

Now from the expression  $C_S(\theta) = \exp(-i\theta D_S) = \exp(-i\theta |S\rangle\langle S|)$  of (2.1) ( $|S\rangle$  is just the solution state  $|x_0\rangle$  here) one knows that the unitary oracle selective diagonal operator  $C_S(\theta)$  selectively acts on only the solution state  $|S\rangle$  without disturbing any other states of the quantum system. When it acts on an arbitrary initial state of the quantum system, one obtains

$$C_S(\theta) : \sum_{x \in \{0,1\}^n} b_x |x\rangle \rightarrow \sum_{x \in \{0,1\}^n, x \neq x_0} b_x |x\rangle + b_{x_0} \exp(-i\theta) |x_0\rangle, \text{ for } |S\rangle = |x_0\rangle. \quad (5.17)$$

It is easy to find from (5.11), (5.15), (5.16), and (5.17) that  $BFSEQ(x_0, \theta)$ ,  $BFSEQ$ ,  $U_o(\theta)$ , and  $C_S(\theta)$  all generate the same unitary transformation of

(5.4). Because  $C_S(\theta) = \exp(-i\theta|S\rangle\langle S|)$  is not involved in any state of the quantum system except the solution state  $|S\rangle = |x_0\rangle$ , it should be considered to be equivalent to the selective black-box functional operation  $BFSEQ(x_0, \theta)$  rather than the usual oracle operation  $U_o(\theta)$  or the black-box functional operation  $BFSEQ$ . As shown above, the usual oracle operation  $U_o(\theta)$  is equivalent to the black-box functional operation  $BFSEQ$ . They are the quantum-parallel functional operations. On the other hand, the unitary oracle selective diagonal operator  $C_S(\theta)$  is equivalent to the selective black-box functional operation  $BFSEQ(x_0, \theta)$  because they are the single functional operations. However, both  $C_S(\theta)$  and  $BFSEQ(x_0, \theta)$  are not equivalent to any one of  $U_o(\theta)$  and  $BFSEQ$ , because the former is the selective functional operations, while the latter is a quantum-parallel functional operation.

In appearance both  $C_S(\theta)$  of (5.17) and  $U_o(\theta)$  of (5.16) generate the same unitary transformation. This made the author explaining misunderstandingly  $C_S(\theta)$  as the faithful representation of  $U_o(\theta)$  in the past. However, as shown above, they are not equivalent to one another at all. Their mathematical-logical meanings are not the same one. The usual oracle operation  $U_o(\theta)$  is a quantum-parallel functional operation and it already takes into account the mathematical-logical meaning that the solution state ( $|x_0\rangle$ ) can be an arbitrary usual computational basis state of the unstructured search space which is just the Hilbert space of the  $n$ -qubit quantum system in a conventional quantum search algorithm. In contrast, the unitary oracle selective diagonal operator  $C_S(\theta)$  with  $|S\rangle = |x_0\rangle$  is a selective functional operation and it is not able to take into account the mathematical-logical meaning that the solution state can be an arbitrary usual computational basis state of the unstructured search space. Consider the fact that the usual oracle operation  $U_o(\theta)$  can characterize essentially the mathematical-logical principle of the unstructured search problem. Then it is clear that the unitary oracle selective diagonal operator  $C_S(\theta)$  with  $|S\rangle = |x_0\rangle$  alone is not able to characterize essentially the same mathematical-logical principle of the search problem. Therefore, while the usual oracle operation  $U_o(\theta)$  can act as the basic building block to construct a conventional quantum search algorithm, the unitary oracle selective diagonal operator  $C_S(\theta)$  with  $|S\rangle = |x_0\rangle$  alone can not be used as the basic building block to construct the *HSSS* quantum search process.

In order that  $C_S(\theta)$  can act as the basic building block to construct the *HSSS* quantum search process the mathematical-logical meaning that the solution state  $|x_0\rangle$  can be an arbitrary usual computational basis state of the unstructured search space must be taken into account in the *HSSS* quantum search process. The *HSSS* quantum search process considers that there are  $N$  candidate solution states for the  $N$ -dimensional unstructured search space. Correspondingly in the unitary oracle selective diagonal operator  $C_S(\theta) = \exp(-i\theta|S\rangle\langle S|)$  the basis state  $|S\rangle$  stands for any one of these  $N$  candidate solution states without limiting to the real solution state  $|x_0\rangle$  only. This really extends the mathematical-logical meaning for  $C_S(\theta)$  and in the meantime  $C_S(\theta)$  is still a selective black-box functional operation for any candidate solution state  $|S\rangle$  (or  $|S\rangle$ ). Now one has  $C_S(\theta) = BFSEQ(x_0, \theta)$  if  $|S\rangle = |x_0\rangle$

and  $C_S(\theta) \neq BFSEQ(x_0, \theta)$  if  $|S\rangle \neq |x_0\rangle$ . These (candidate) unitary oracle selective diagonal operators altogether are considered as a single entity. This is crucial for the search-space dynamical reduction in the *HSSS* quantum search process. With this extended mathematical-logical meaning the unitary oracle selective diagonal operator  $C_S(\theta)$  is able to represent faithfully the black-box functional operation *BFSEQ* of (5.4).

There is an essential difference between the unitary oracle selective diagonal operator  $C_S(\theta)$  with the real solution state  $|S\rangle = |x_0\rangle$  and the one with a candidate solution state  $|S\rangle \neq |x_0\rangle$ . When the former acts on an initial state of a quantum system, the generated state is still in the quantum system. In contrast, when the latter is applied to the same initial state in the unstructured search space, the generated state is merely a candidate state and it is not in the quantum system (See the section 3 above).

The realization that  $C_S(\theta)$  can represent faithfully the black-box functional operation *BFSEQ* can not be carried out alone in the Hilbert space of a quantum system. The reason for this is that the realization is involved in the mathematical-parallel computation (or operation) that does not obey the superposition principle in quantum mechanics. (Here in effect mathematical parallel computation is classical parallel computation.) In principle the realization can be carried out alone in the math Hilbert space of the search problem (See this section above and also the section 3) as a mathematical-parallel computation is allowed in the math Hilbert space, but it has nothing to do with a quantum system and consequently it can not be implemented practically in quantum computation. The unique possibility is that the realization is carried out in both the Hilbert space of the quantum system and the math Hilbert space of the search problem, as shown in the section 2 above.

The decomposition formula (5.14) of quantum-parallel operation provides the answer for why the unitary oracle selective diagonal operator  $C_S(\theta)$  is reducible. The black-box functional operation *BFSEQ* of (5.4) is a quantum-parallel operation and it is irreducible as a whole. It is known from (5.14) that *BFSEQ* is a consecutive sequence of these  $2^n$  selective black-box functional operations  $\{BFSEQ(y, \theta)\}$  and  $BFSEQ(x_0, \theta)$  is merely one of these  $2^n$  number of  $\{BFSEQ(y, \theta)\}$ . Then the formula (5.14) shows that although the black-box functional operation *BFSEQ* as a whole is irreducible, the selective black-box functional operation  $BFSEQ(x_0, \theta)$  is allowed to be reducible. Consider the fact that the unitary oracle selective diagonal operator  $C_S(\theta)$  with  $|S\rangle = |x_0\rangle$  is equivalent to  $BFSEQ(x_0, \theta)$ . Then one can conclude that  $C_S(\theta)$  is allowed to be reducible.

The quantum-computing speedup theory [1] then points out that whether or not the unitary oracle selective diagonal operator  $C_S(\theta)$  can be reduced is determined by the fundamental quantum-computing resource, i.e., the symmetric structure of the Hilbert space of a composite quantum system and moreover, if it can, then its reduction process is performed in the frame of unitary quantum dynamics.

The practical implementation (or simulation) for  $C_S(\theta)$  needs to employ two computing machines, one of which is the quantum-physical computing ma-

chine (e.g., an  $n$ -qubit quantum system) and another the quantum-computing math machine which owns the math Hilbert space and is mathematical. The quantum-physical computing machine computes the black-box functional operation  $BFSEQ$  of (5.4) with the real solution state  $|x_0\rangle$ , while the quantum-computing math machine computes the black-box functional operation  $BFSEQ$  with any candidate solution state  $|S\rangle \neq |x_0\rangle$  and executes the mathematical-parallel functional operation for  $BFSEQ$  in the math Hilbert space. All these computations of the black-box functional operation  $BFSEQ$  with the sequence (5.2) are efficient (in experiment or in the sense of theory) as long as the reversible Boolean functional operation  $U_f$  of (5.1) can be realized efficiently. Therefore,  $C_S(\theta)$  can be efficiently implemented (or simulated) practically as long as  $U_f$  can be realized efficiently or more directly as long as the black-box functional operation  $BFSEQ$  of (5.4) can be realized efficiently.

The main mathematical-logical differences between  $U_o(\theta)$  and  $C_S(\theta)$  are summarized as follows. Both  $U_o(\theta)$  and  $C_S(\theta)$  can represent faithfully the black-box functional operation  $BFSEQ$  of (5.3) or (5.4), but they use different Hilbert spaces. The usual oracle operation  $U_o(\theta)$  uses only the Hilbert space of the quantum system to realize this faithful representation, while  $C_S(\theta)$  needs to use explicitly both the Hilbert space of the quantum system and the math Hilbert space of the search problem. Of course, it also may be thought that in the usual oracle operation  $U_o(\theta)$  both the Hilbert spaces are coincident completely with one another, as shown in the section 2 above. In the Hilbert space of the quantum system the usual oracle operation  $U_o(\theta)$  is a quantum parallel operation and irreducible in the mathematical-logical meaning of unstructured search, while  $C_S(\theta)$  is a single black-box functional operation and does not have the mathematical-logical meaning of quantum parallel operation and moreover, it is allowed to be reducible in the mathematical-logical meaning of unstructured search. Besides these differences, in the math Hilbert space  $C_S(\theta)$  may be considered as a mathematical-parallel black-box functional operation as a whole due to that the real solution state is unique and deterministic.

The difference of mathematical-logical meaning between  $C_S(\theta)$  and  $U_o(\theta)$  is helpful for understanding better the essential difference of the quantum-searching speedup mechanism between the  $HSSS$  quantum search process and a conventional quantum search algorithm. Since the usual oracle operation  $U_o(\theta)$  is irreducible, there is the square speedup limit on a conventional quantum search algorithm. Since  $U_o(\theta)$  works only in the Hilbert space of the quantum system, it may be considered as a  $QM$  unitary operator. This results in that it can not change the quantum-state difference between any pair of quantum states of the quantum system. In contrast, since the unitary oracle selective diagonal operator  $C_S(\theta)$  is allowed to be reducible, the search-space dynamical reduction becomes feasible as long as there is the fundamental quantum-computing resource. Since  $C_S(\theta)$  works in both the Hilbert space of the quantum system and the math Hilbert space of the search problem, its behavior can not be completely described alone by the quantum-mechanical principles. Instead, it is described completely by both the quantum-mechanical principles and the mathematical-logical principle of the search problem. This results in that though

it is a unitary operator, it is able to change the quantum-state difference between any pair of quantum states that belong to the two Hilbert spaces, respectively. Due to these two reasonings the *HSSS* quantum search process with the basic building block  $C_S(\theta)$  may not be constrained by the square speedup limit of a conventional quantum search algorithm.

With the basic building block  $C_S(\theta)$  the *HSSS* quantum search process sets up its own strategy to find the real solution state among these  $N$  (here  $N = 2^n$ ) candidate solution states (or equivalently distinguish the real solution state from the other  $N - 1$  candidate solution states). As well known before, the strategy consists of the search-space dynamical reduction and the unitary dynamical state-locking process and its inverse process. The theoretical basis behind the strategy is the interaction between the quantum-physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetrical structure and property) and the mathematical-logical principle of the search problem. The *HSSS* quantum search process mainly consists of the two consecutive steps. The first step is the efficient search-space dynamical reduction in the frame of unitary quantum dynamics [2]. It eliminates the unstructured search space of the search problem. This step becomes feasible because the unitary oracle selective diagonal operator  $C_S(\theta)$  may be reducible and there is the fundamental quantum-computing resource in the  $n$ -qubit quantum system used to solve the search problem. The second step is the exponential quantum-state-difference amplification [36] which in principle is already described in detail in the previous sections 3 and 4 in the paper.

Below explains the quantum-computing speedup mechanism for the search-space dynamical reduction. It must be pointed out that without the search-space dynamical reduction the computational complexity of an unstructured search problem can not be essentially changed even if one employs the unitary oracle selective diagonal operator  $C_S(\theta)$  as the basic building block to construct any quantum search algorithm to solve the search problem. Therefore, it is still hard to extract the information of the component states of the solution state for any quantum search algorithm without the search-space dynamical reduction. Such quantum search algorithm still must be subject to the square speedup limit of a conventional quantum search algorithm. On the other hand, from the mathematical-logical viewpoint  $C_S(\theta)$  is a mathematical-parallel black-box functional operation in the unstructured search space (i.e., the math Hilbert space) with exponentially large dimension  $N = 2^n$ . It owns exponentially many mathematical-parallel computational paths. The extraction of the information of the component states of the solution state is a hard task, since one needs first to compute these detailed computational paths and then treat these computational results and all these are hard things.

When the exponentially large unstructured search space is cancelled, all these exponentially many mathematical-parallel computational paths are accordingly eliminated. That is, when the exponentially large unstructured search space is reduced dynamically to a polynomially small subspace, the black-box functional operation ( $C_S(\theta)$ ) with exponentially many computational paths is replaced with the one with only polynomially many computational paths. Now

it could become feasible to compute a few or even a polynomially many computational paths. Therefore, the search-space dynamical reduction greatly speeds up to extract the information of the component states of the solution state. Moreover, due to the search-space dynamical reduction the exponential quantum-state-difference amplification [36] becomes feasible for the quantum states that carry the information of the component state. These are the contribution of the search-space dynamical reduction to the exponential quantum-computing speedup in the *HSSS* quantum search process.

Finally outlines the solution information flow over the *HSSS* quantum search process. The information flow is one-way. It starts from the reversible Boolean functional operation that initially loads the solution information of an unstructured search problem and may be considered as the initial information source in the *HSSS* quantum search process. The solution information is first delivered to the black-box functional operation (which also could be considered as the initial information source). Then it is transferred in form to the unitary oracle selective diagonal operator. Up to this stage the solution information is complete. Then by the efficient search-space dynamical reduction in the frame of unitary quantum dynamics the component of the solution information, i.e., the information of the component state of the solution state is output to the information-carrying unitary operator. This step creates the information source of the component state of the solution state. After the exponential (unitary) QUANSDAM process, the information of the component state is extracted by the quantum measurement.

## 6. Discussion

A UNIDYSLOCK (or QUANSDAM) process is the characteristic quantum computational process (or sub-process) of the quantum-computing speedup theory. It obeys the unitary quantum dynamics, but it is able to change the quantum-state difference between a pair of quantum states. This characteristic property of a UNIDYSLOCK (or QUANSDAM) process is original from the fundamental interaction between the quantum-physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) and the mathematical-logical principle that a computational problem obeys. It is shown in the paper that any unitary dynamical process in quantum mechanics can not change the quantum-state difference. It is also shown that no quantum-state effect can cause the quantum-state-difference varying during a UNIDYSLOCK (or QUANSDAM) process in a quantum system. This conclusion is universal in the frame of unitary quantum dynamics for any quantum computational process that employs a UNIDYSLOCK (or QUANSDAM) process (via the unstructured quantum search process) to realize its quantum-computing speedup. A UNIDYSLOCK (or QUANSDAM) process therefore is able to distinguish the quantum-computing speedup theory from any conventional quantum computational theory. The conventional quantum computational theory based on the quantum parallel principle [11a] considers that a quantum-computing speedup is essentially original from the quantum-state effects of a quantum system. Here the quantum-state effects include the quantum-



state superposition, coherence interference, entanglement and nonlocal effect, correlation and so on, and among these quantum-state effects the quantum entanglement and nonlocal effect is considered as the essential one. Then this really means that in the conventional quantum computational theory there is not an essential contribution for a UNIDYSLOCK (or QUANS DAM) process to the quantum-computing speedup of any reversible (or unitary) quantum computational process. Therefore, whether or not a UNIDYSLOCK (or QUANS DAM) process can make an essential contribution to the quantum-computing speedup of a reversible or unitary quantum computational process distinguishes strictly the quantum-computing speedup theory from any conventional quantum computational theory in the frame of unitary quantum dynamics.

Can the conventional quantum computational theory employ a UNIDYSLOCK (or QUANS DAM) process to realize a quantum-computing speedup? There is no doubt that a UNIDYSLOCK (or QUANS DAM) process makes no contribution to the quantum-computing speedup of a conventional quantum computation if the quantum computation does not contain any UNIDYSLOCK (or QUANS DAM) process. However, even if a conventional quantum computation that is reversible or unitary contained any UNIDYSLOCK (or QUANS DAM) process, the contribution of the UNIDYSLOCK (or QUANS DAM) process to the quantum-computing speedup of the conventional quantum computation would be secondary or negligible. The reason for this is that if the contribution of the UNIDYSLOCK (or QUANS DAM) process were dominating, then the one of the quantum-state effects to the quantum-computing speedup would be secondary or negligible. There are only two nontrivial physical processes that can change the quantum-state difference between a pair of quantum states. One of which is a UNIDYSLOCK (or QUANS DAM) process. It is unitary. Another is a non-equilibrium irreversible process (which could contain the usual quantum measurement). It is not unitary. Then the only possibility for a conventional quantum computational process to employ a quantum-state-difference amplification to realize its quantum-computing speedup is that the quantum-state-difference amplification process must be irreversible! As shown in the section 3 above, without unitarity a quantum-state-difference amplification process, e.g., a UNIDYSLOCK (or QUANS DAM) process could become trivial in quantum computation. It also suffers from the energy dissipation problem of irreversibility (See, e.g., Ref. [20]) in quantum computation. Even for the case that the quantum-state-difference amplification process is a non-equilibrium irreversible process one can not yet conclude that the quantum-computing speedup is original from the quantum-state effects of the quantum system, because in that case it is still possible that the quantum-state-difference amplification could be original from the irreversible dynamical process itself and it could have nothing to do with any quantum-state effect of the quantum system. Physical world in nature is fundamentally reversible. Therefore, although apparently the unitary quantum dynamics often could not be obeyed strictly in nature, the quantum-computing speedup theory [1] considers that it is not able to deny the fact that the unitary quantum dynamics is a fundamental principle in nature. Thus, the theory thought that *even such a process as a non-equilibrium irreversible pro-*

*cess in nature that does not obey apparently the unitary quantum dynamics is governed by the unitary quantum dynamics.*

Physicalization of a mathematical-logical functional operation is a universal phenomenon in classical reversible computation [20] and in conventional quantum computation [9, 10, 11]. It is not only related to whether or not a mathematical-logical function is computable in a quantum system but it also could affect the quantum-computing speedup mechanism. The physicalization of a reversible functional operation is simply described as follows. When it is realized in a physical system, a reversible functional operation in mathematics becomes a pure reversible (or unitary) physical process of the system. In the meantime, the function-operational space in mathematics is mapped one-to-one onto the physical state space of the system. Then the latter acts as the former in the realization of the functional operation in the physical system. Thus, both the function-operational space and the physical state space are the same one in the realization. Both the spaces may have their own symmetric structures and properties, respectively. But because they are the same one, the symmetric structure and property of the physical state space may be washed out and there is no consideration of the interaction between the two spaces. These are the common basic characteristic features for the physicalization of a reversible functional operation in the classical reversible computation and the conventional quantum computation.

The physicalization of a reversible functional operation is necessary for the conventional quantum computational theory based on the quantum parallel principle [11a]. Though a huge number of the functional states can be computed simultaneously by a quantum parallel functional operation, it is hard to obtain the desired computational result from the final state of the operation. The theory then employs a variety of quantum-state effects to extract the desired computational result from the final state. Though these quantum-state effects could be constrained by the mathematical symmetrical structure and property of the computational problem to be solved, this constraint (or its influence) is considered to be inessential or secondary in the quantum-computing speedup mechanism based on the quantum parallel principle [11a]. Therefore, it is generally considered in the theory that a quantum-computing speedup is essentially original from these quantum-state effects themselves. The latter mainly includes the quantum-state superposition, coherence interference, entanglement and nonlocal effect, and correlation. Furthermore, the quantum entanglement and nonlocal effect is generally considered as the essential one among these quantum-state effects. However, these quantum-state effects between any two or more different (component) functional states of the final state of the quantum parallel functional operation could exist only when these functional states are real quantum states of the quantum system. Imaging that one component functional state is a real quantum state, while another is an unphysical state. Then it is impossible in quantum mechanics that these two functional states can generate the quantum-state effects such as the quantum-state superposition, coherence interference, entanglement and nonlocal effect, and correlation and so on. Therefore, the quantum parallel operation of a function must be

physical. That is, the physicalization of a mathematical-logical functional operation is necessary in the conventional quantum computation. This also indicates clearly that there is not a separate math Hilbert space in the conventional quantum computation. The physicalization of a reversible functional operation could make the quantum-state effects of the functional states dominating in the quantum-computing speedup mechanism of the conventional quantum computational theory based on the quantum parallel principle. But the symmetric structure of the Hilbert space of the quantum system (i.e., the fundamental quantum-computing resource) may be washed out.

The quantum-computing speedup theory [1] does not think that these quantum-state effects can be responsible for an essential quantum-computing speedup. The difficulties and limitations of the quantum-computing speedup mechanism based on the quantum parallel principle [11a] have been analyzed and discussed in detail in Ref. [1]. Here these will not be further discussed.

One important consequence for the physicalization of a reversible functional operation in conventional quantum computation is that the quantum-state difference between any pair of quantum states can not be changed by a reversible functional operation, because the reversible functional operation is treated as a pure  $QM$  unitary operator in conventional quantum computation after it is physicalized in the quantum system. Another is that in conventional quantum computation the fundamental quantum-computing resource of a quantum system is not exploited to speed up a quantum computation.

In contrast, in the quantum-computing speedup theory the reversible functional operation of a computational problem is originally responsible for the quantum-state-difference varying in a UNIDYSLOCK (or QUANSDAM) process and the fundamental quantum-computing resource is responsible for an essential quantum-computing speedup.

For a long time the physicalization of a reversible functional operation or more generally the physicalization of computing has been mainstream in quantum computational science. This can be shown by the Church-Turing principle [11a]: "Every finitely realizable physical system can be perfectly simulated by a universal model computing machine operating by finite means". The physicalization of computing is the computing version of information physicalization. While information physicalization seems to be quite reasonable, the largest obstacle facing the computing physicalization comes from the fact that computing must obey the mathematical-logical principle. Quantum computing can not be described completely by the quantum-physical laws alone and the mathematical-logical principle of a computational problem to be solved can not be disregarded in the quantum-computing speedup mechanism. This is consistent with the spirit of the quantum-computing speedup theory [1].

The quantum-computing speedup theory considers that the fundamental interaction between the quantum-physical laws (i.e., the unitary quantum dynamics and the Hilbert-space symmetric structure and property) and the mathematical-logical principle that a computational problem obeys is the origin of an essential quantum-computing speedup. Then in the theory both the math Hilbert space of a computational problem and the Hilbert space of the

quantum system used to solve the computational problem must be treated separately. One important reason why both the Hilbert spaces are treated separately in the theory is that one needs to realize the search-space dynamical reduction [5, 6, 1, 2]. In contrast, both the Hilbert spaces are the same one in conventional quantum computation due to the physicalization of computing. The math Hilbert space of a computational problem is a fundamental concept in the quantum-computing speedup theory. The concept is characteristic for the theory. It is not owned by the conventional quantum computational theory. Due to the math Hilbert space the manner to realize a reversible functional operation in the quantum-computing speedup theory is essentially different from the one in conventional quantum computation. A mathematical-parallel functional operation is the characteristic manner in the quantum-computing speedup theory. In contrast, a quantum-parallel functional operation is the characteristic manner in the conventional quantum computational theory based on the quantum parallel principle. Due to the interaction between the math Hilbert space and the Hilbert space of the quantum system a reversible functional operation is able to change the quantum-state difference between a pair of quantum states in the quantum-computing speedup theory. This leads to that there is a UNIDYS-LOCK (or QUANS DAM) process which is characteristic for the theory. Of course, the separate treatment for the two Hilbert spaces is not the purpose for the theory. Instead, the theoretical purpose is to set up the interaction between the two Hilbert spaces in the frame of unitary quantum dynamics so that by the interaction the fundamental quantum-computing resource of the quantum system can be harnessed to speed up (directly or indirectly) a quantum computation. In contrast, there is not the interaction in conventional quantum computation, because both the Hilbert spaces are the same one due to the physicalization of computing. Thus, there is no exploitation of the fundamental quantum-computing resource to speed up a quantum computation in conventional quantum computation.

Here one must distinguish the fundamental quantum-computing resource from the quantum-state effects in quantum-computing speedup mechanism. The former is employed by the quantum-computing speedup theory [1] to speed up a quantum computation, while the latter is considered by the conventional quantum computational theory [11a] as the origin of a quantum-computing speedup. It is well known in quantum mechanics that the Hilbert space of a composite quantum system is formed of all the quantum states of the quantum system. The symmetric structure and property of the Hilbert space therefore is the global property of the quantum system. It is independent on any concrete quantum state of the quantum system. In the quantum-computing speedup theory the Hilbert-space symmetric structure is considered as the fundamental quantum-computing resource and it is responsible for an exponential quantum-computing speedup. It is also well-known in quantum mechanics that two (appropriate) quantum states (or particles) of the quantum system alone are sufficient to generate the quantum-state effects such as the quantum-state superposition, coherence interference, entanglement and nonlocal effect, and correlation and so on. These quantum-state effects are generally considered as the partial (or

partite) property of the quantum system. The conventional quantum computational theory considers that a quantum-computing speedup is original from these quantum-state effects.

A reversible functional operation is the core of a quantum algorithm to solve a computational problem in quantum computation. A computational problem usually could be characterized completely by using the reversible functional operations, when it is solved in quantum computation. In this paper the author does not study how to realize a reversible functional operation with a universal set of one- and two-qubit quantum logic gates (or more complex gate sets) in a quantum system (See Ref. [35] and therein for this research topic). Instead, what the author studies is that the reversible functional operation of a computational problem is directly considered as the starting point of research subject. Why the quantum-computing speedup theory emphasizes the importance of a computational problem? It is well known that what standard quantum mechanics treats is atomic, molecular, nuclear system and so on. Correspondingly, what the quantum-computing speedup theory treats is any concrete computational problem. Each computational problem obeys its own mathematical-logical principle and has its own mathematical symmetric structure.

A computational problem usually could be characterized completely by the functional operations, when it is solved in computation. Therefore, these functional operations usually could be used as the basic building blocks of an algorithm to solve the computational problem in computational science, here the algorithm may be classically irreversible, classically reversible, or quantum. These functional operations that can characterize completely the computational problem obey obviously the specific mathematical-logical principle which could consist of a series of mathematical-logical rules or conditions imposed by the computational problem. One may say that the mathematical-logical principle is owned only by the computational problem. It is well-known that a functional operation may be carried out in a Turing machine in mathematical computation and it also may be performed in a physical computing machine (i.e., a physical system) in physical computation. However, the mathematical-logical principle of a computational problem is essentially independent of any computing machine which may be mathematical or physical, because a computational problem is independent of any computing machine. For example, consider the problem that one wants to determine whether an integer  $p$  is prime or composite. As well known in number theory, (i) if the integer  $p$  is prime, then it can be divided only by one and itself; (ii) if  $p$  is not prime, then it must be a product ( $st$ ) of at least two integers  $s$  and  $t$  with  $1 < s, t < p$ . The two properties (i) and (ii) form the essential components of the mathematical-logical principle of the present problem. Any one of these two properties must be obeyed no matter which computing machine one uses to solve the present problem. Of course, statement or expression for the mathematical-logical principle of the computational problem could be different from machine to machine, but its essence is the same!

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## Appendix A: The information-carrying unitary propagators generated approximately in a single-atom system

The spatially-selective and internal-state-selective triggering pulse [15, 31] may be used to realize the *IC* internal-state-dependent unitary momentum-displacement propagator  $U_p^{ic}(a_m^s)$  of (4.3) in a single-atom system. Here it is not considered whether this realization is efficient or not. In Refs. [15, 31] the triggering pulse is constructed approximately with the help of the Trotter-Suzuki method mainly [16] and by using the interaction [29, 30] between the internal motion and the COM motion of the single-atom system. Inspired by the construction of the triggering pulse here a simple method is proposed to realize approximately the *IC* unitary propagator  $U_p^{ic}(a_m^s)$  of (4.3) in a single-atom system. In this method a Baker-Campbell-Hausdorff (BCH)-type operator identity is first obtained that is used to generate exactly the unitary operator  $\exp(-\tau^2[A, B])$  from the unitary operators  $\exp(-iA\tau)$  and  $\exp(-iB\tau)$  with the Hermitian operators  $A$  and  $B$ . This BCH-type operator identity may be explicitly expressed as

$$\exp(-iA\tau)\exp(-iB\tau)\exp(+iA\tau)\exp(+iB\tau) = \exp(-\tau^2[A, B]) + O_p(\tau^3) \quad (\text{A1})$$

where the error operator  $O_p(\tau^3)$  is exactly written as

$$O_p(\tau^3) = \int_0^\tau d\lambda \exp(-i\lambda A) \exp(-i\lambda B) Q_R(\lambda) \exp(-(\tau^2 - \lambda^2)[A, B]) \quad (\text{A2})$$



with the operator  $Q_R(\lambda)$  given by

$$\begin{aligned}
Q_R(\lambda) = & -2\lambda \exp(i\lambda A) \int_0^\lambda ds \{ \exp(isB) [[A, B], iB] \exp(-isB) \\
& + \exp(-isA) [[A, B], iA] \exp(isA) \} \exp(i\lambda B) \\
& + \int_0^\lambda ds \int_0^s ds' \{ \exp(is'B) [[A, B], iB] \exp(-is'B) \\
& + \exp(is'A) [[A, B], iA] \exp(-is'A) \} \exp(i\lambda A) \exp(i\lambda B). \quad (A3)
\end{aligned}$$

This operator identity can be proven by using the operator algebra method similar to those in Ref. [16b].

On the basis of the operator identity (A1) one may construct the  $IC$  unitary propagator  $\exp(-\tau^2[A, B])$  with the suitable Hermitian operators  $A$  and  $B$ . Here one or two of the operators  $A$  and  $B$  must carry the solution information ( $a_m^s$ ) of the search problem. An example is given below to show how to construct the  $IC$  unitary propagator.

Case (1) : The single-atom system is a free atom. Suppose that in the rotating frame the interaction between the atomic internal motion and COM motion is simply given by  $H_I = -KxI_{mx}$ , where  $I_{mx}$  is the  $x$ -component spin-1/2 operator of the atomic internal motion. The interaction could be generated by the external electromagnetic field or static electric (and/or magnetic) field<sup>9</sup>. Here for simplicity it is not discussed in detail how to generate the interaction. Then the Hamiltonian of the atom in the external field is given by  $H_A = \frac{1}{2m}p^2 + H_a + H_I$ , where in the rotating frame the atomic internal Hamiltonian  $H_a$  is zero under on-resonance condition. Now in the operator identity (A1) one may set the operator  $A = a_m^s \theta_m I_{my} / \tau$  and  $B = H_A / \hbar = (\frac{1}{2m}p^2 + H_I) / \hbar$ . Then it can prove that the commutator  $[A, B] = (ia_m^s \theta_m / \tau)(K/\hbar)I_{mx}$ .

Case (2) : The single-atom system is a single atom in an external harmonic potential field. Here still suppose that in the rotating frame the interaction between the atomic internal motion and COM motion is simply given by  $H_I = -KxI_{mx}$ . Then the total atomic Hamiltonian is given by  $H_A = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + H_a + H_I$ , where  $H_a = 0$  under on-resonance condition in the rotating frame. Now setting  $A = a_m^s \theta_m I_{my} / \tau$  and  $B = H_A / \hbar = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + H_I$  one still obtains the same commutator  $[A, B] = (ia_m^s \theta_m / \tau)(K/\hbar)I_{mx}$ .

In the above commutator  $[A, B]$  the spin operator  $I_{m\lambda}$  ( $\lambda = x, y, z$ ) acts on only the internal state of the single-atom system, while the coordinate operator  $x$  acts on only the COM motional state of the system. From the commutator it follows that the unitary operator  $\exp(-\tau^2[A, B])$  on the right-hand (RH) side of (A1) is given by

$$\exp(-\tau^2[A, B]) = \exp(-ia_m^s(\theta_m \tau)(K/\hbar)I_{mx}). \quad (A4)$$

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<sup>9</sup>See, for example, B. W. Shore, *The theory of coherent atomic excitation*, Vol.1, Section 2.8 & 3.3, Wiley, New York, 1990; J. D. Jackson, *Classical electrodynamics*, 2nd., Wiley & Son, New York, 1975; L. Allen and J. H. Eberly, *Optical resonance and two-level atoms*, Dover, New York, 1987

This unitary operator carries the solution information ( $a_m^s$ ). Thus, it is really an information-carrying unitary operator, i.e., the internal-state-dependent unitary momentum-displacement propagator  $U_p^{ic}(a_m^s)$  of (4.3). On the left-hand (LH) side of (A1) the unitary operator  $\exp(\pm iA\tau)$  is given by

$$\exp(\pm iA\tau) = \exp(\pm ia_m^s \theta_m I_{my}). \quad (\text{A5})$$

By comparing it with (2.9) one sees that this unitary operator is in form just the basic *IC* unitary operator of (2.9). Since the operator  $B = H_A/\hbar$  is the Hamiltonian of the single-atom system, it does not carry the solution information ( $a_m^s$ ). Then the unitary operator  $\exp(\pm iB\tau)$  on the LH side of (A1) is the *QM* unitary operator. These show that the LH side of (A1) is the unitary sequence of the basic *IC* unitary operators and the *QM* unitary operators, while the RH side of (A1) is the *IC* unitary propagator  $\exp(-\tau^2[A, B])$  in addition to the error operator  $O_p(\tau^3)$ . Therefore, the operator identity (A1) could be used to construct the *IC* unitary propagator  $U_p^{ic}(a_m^s)$  of (4.3) by using the basic *IC* unitary operators of (A5) and the *QM* unitary operators in the single-atom system. Of course, the pre-condition for this is that the error operator  $O_p(\tau^3)$  on the RH side of (A1) can be neglected.

If one deletes the solution information ( $a_m^s$ ) in the operator  $A$  above, then the unitary propagator  $\exp(-\tau^2[A, B])$  generated by (A1) is just the internal-state-selective triggering pulse [31, 15] and it could be further developed as a spatially-selective and internal-state-selective triggering pulse for a wave-packet motional state such as a Gaussian wave-packet motional state of the single-atom system [15].

On the RH side of (A1) the *IC* unitary operator  $\exp(-\tau^2[A, B])$  is the main term, while the error operator  $O_p(\tau^3)$  is usually a secondary term. The latter is dependent on the parameter  $\tau$  and the operators  $A$  and  $B$ . In a general case the operator identity (A1) could be used to generate the lowest-order approximate unitary sequence for  $\exp(-\tau^2[A, B])$ . Then by starting from the lowest-order approximation one could employ the Trotter-Suzuki method [16a] to generate a higher-order approximate unitary sequence for  $\exp(-\tau^2[A, B])$ .

As a simple example, from the operator identity (A1) one may further obtain the operator identity:

$$\begin{aligned} & (\exp(-iA\tau/n) \exp(-iB\tau/n) \exp(+iA\tau/n) \exp(+iB\tau/n))^{n^2} \\ & = \exp(-\tau^2[A, B]) + O_p(n^2 (\tau/n)^3). \end{aligned} \quad (\text{A6})$$

Here in theory the exact expression for the error operator  $O_p(n^2 (\tau/n)^3)$  may be obtained from the operator identity (A1). If in (A6) one takes  $n \rightarrow \infty$ , then one has

$$\begin{aligned} & \lim_{n \rightarrow \infty} (\exp(-iA\tau/n) \exp(-iB\tau/n) \exp(+iA\tau/n) \exp(+iB\tau/n))^{n^2} \\ & = \exp(-\tau^2[A, B]). \end{aligned} \quad (\text{A7})$$

Note that  $\exp(-iA\tau/n)$  and  $\exp(-iB\tau/n)$  are unitary. Then this formula holds even when the Hermitian operators  $A$  and  $B$  are unbounded (See, for example, T. F. Jordan, *Linear operators for quantum mechanics*, Chapt. VII, Dover, New York, 2006). It shows that when the number  $n$  is sufficiently large, the error operator  $O_p(n^2(\tau/n)^3)$  in (A6) can be neglected, and the *IC* unitary propagator  $\exp(-\tau^2[A, B])$  can be generated approximately by the unitary sequence on the LH side of (A6) which consists of the basic *IC* unitary operators of (A5) and the *QM* unitary operators  $\{\exp(\pm iB\tau/n)\}$ . Both the formulae (A6) and (A7) do not consider the computational complexity for the approximate realization of the *IC* unitary propagator  $\exp(-\tau^2[A, B])$ . Whether or not the *IC* unitary propagator  $\exp(-\tau^2[A, B])$  can be realized efficiently by the unitary sequence on the LH side of (A6) is largely dependent on the error operator  $O_p(n^2(\tau/n)^3)$ . Here there is not a rigorous mathematical proof to show whether this realization is efficient or not.

Suppose now that  $|0\rangle$  and  $|\Psi(x, t_0)\rangle$  are the internal state and the COM motional state of the single-atom system, respectively, where  $|0\rangle$  is the eigenstate of the spin-1/2 operator  $I_{mz}$  of the atomic internal motion and the eigenvalue equation for the operator  $I_{mz}$  is given by  $I_{mz}|0\rangle = \frac{1}{2}|0\rangle$ . Now applying both sides of (A6) to the initial product state  $|\Psi(x, t_0)\rangle|0\rangle$  of the single-atom system one obtains the QUANSDAM process:

$$\begin{aligned} & (\exp(-iA\tau/n) \exp(-iB\tau/n) \exp(+iA\tau/n) \exp(+iB\tau/n))^n |\Psi(x, t_0)\rangle|0\rangle \\ &= \exp(-\tau^2[A, B]) |\Psi(x, t_0)\rangle|0\rangle + O_p(n^2(\tau/n)^3) |\Psi(x, t_0)\rangle|0\rangle. \end{aligned} \quad (\text{A8})$$

Here using (A4) one finds that

$$\exp(-\tau^2[A, B]) |\Psi(x, t_0)\rangle|0\rangle = \exp(-ia_m^s p_0 x / \hbar) |\Psi(x, t_0)\rangle|0\rangle \quad (\text{A9})$$

where  $p_0 = \frac{1}{2}K\tau\theta_m$ . The *IC* unitary operator  $\exp(-ia_m^s p_0 x / \hbar)$  in (A9) is an *IC* unitary momentum-displacement propagator. If now the initial motional state  $|\Psi(x, t_0)\rangle$  is chosen as a momentum eigenfunction of the single-atom system, then the unitary process (A9) is really the phase-based QUANSDAM process of (4.4). In this case, if the error term on the RH side of (A8) can be neglected, then the LH side of (A8) is really a phase-based QUANSDAM process.

Since an ideal momentum eigenfunction is hard to prepare, one could employ other motional state  $|\Psi(x, t_0)\rangle$  than a momentum eigenfunction to realize the QUANSDAM process of (A8). For example, one could choose a Gaussian wave-packet motional state to act as  $|\Psi(x, t_0)\rangle$  to realize the QUANSDAM process. In this case the error term on the RH side of (A8) may be calculated exactly in theory, although the calculation is not easy at all when the number  $n$  is large. In comparison with the QUANSDAM process of (A7) the QUANSDAM process of (A8) in performance could be greatly improved by a Gaussian wave-packet motional state [15]. So far there is not yet a rigorous mathematical proof to show whether or not the QUANSDAM process of (A8) can achieve an exponential quantum-state-difference amplification.